Reviewers' comments:

Reviewer #1 (Remarks to the Author):

In the paper entitled: “General Theory and Observation of Cherenkov Radiation Induced by Multimode Solitons”, the authors demonstrate that spatiotemporal waves propagating in GRIN and step multimode fibers generate complex dispersive waves. These broadband dispersive waves result from the decay of multimode solitons and can be generated in both the normal and anomalous dispersion domains. These dispersive waves, well known in single mode fiber, are here described for multimode fibers. The authors provide also a theory capable of explaining the dispersive wave dynamics.

This very interesting work is quite similar than the one published by the same group in the paper: “Wright, L. G., Wabnitz, S., Christodoulides, D. N. & Wise, F. W. Ultrabroadband Dispersive Radiation by Spatiotemporal Oscillation of Multimode Waves. Phys. Rev. Lett. 115, 223902 (2015).” and referenced as [18] in the paper. Unfortunately this additional work does not provide huge experimental novelty on dispersive wave generation in multimode fibers. As mentioned by the authors, the developed theory of the paper could elucidate some of the trends in the DW spectrum already published in ref [18].

The theory developed in this paper is also not a new and unpublished approach and is mainly based one the work provided by Crosignani and Di Porto (Ref 37 and 38). In these conditions the paper remains interesting to understand more the DW generation in complex multimode fibers.

Furthermore the authors make statements which should be demonstrated and verified experimentally.
Line 207, the authors claim that: “ According to Eq. (7), to populate the far reaching DW lines, the beam at the soliton fission point should involve a significant portion of the total energy in higher-order modes”, Can the authors provide experimental proof for this claim? Which modes are involved and how many? What means “significant portion”?

Can the authors give the excited modes of the fiber during the coupling of the energy and measure the spatial content of the solitons? This would provide an irrefutable proof of the constitution of multimodal solitons and would allow better positioning of theoretical and numerical approaches.

Can the author provide experimental proof of the modal constitution of dispersive waves in normal and abnormal dispersion regimes?

Reviewer #2 (Remarks to the Author):

The article by Eftekhar et al., presents a comprehensive study on the observation of Cherenkov Radiation in the multimode solitonic system in the context of supercontinuum generation in multimode fibers. Though the topic of Supercontinuum generation in fibers is quite well established, however, the current understanding and the accurate modeling of the dispersive wave generation in the multimode fiber system is far from being complete, owing to the complex nature of this system. In this article, the authors developed a theoretical background to predict the nature of the dispersive wave radiations and substantiated their analytical results through accurate numerical modeling and well thought out experiments. I have a favorable opinion about the manuscript, as I believe this present work contains enough new results interesting to the broad readership of the Optics community and beyond. So, I am happy to recommend the acceptance of the manuscript. However, the author should address the following comments.

From my understanding, the analytical part of the work is a rather important result and I see only very little discussion on that. I would urge the authors to discuss a bit more in detail the analytical solution. Is that Eq.(1) a derived solution pertaining to Eq.(2) or just an assumed standard soliton solution?, If derived, what technique was used to derive it? As there are multiple methods available to arrive at analytical solutions to complex nonlinear equations like Eq.(2).

I have noticed that polarization characteristics are underestimated, I appreciate it if the authors could comment on it.

To conclude, I am happy to recommend the paper for publication provided if the authors make necessary revisions taking into account the above suggestions.

Reviewer #3 (Remarks to the Author):

The authors have derived a general phase matching condition (PMC) for the generation of dispersive radiation from the multimode solitons in step-index as well as the graded-index (GRIN) fibers. Further, they have simplified that PMC for the specific case of the GRIN fibers where the propagation constants of different mode-groups are equally spaced. They have shown a good agreement between theory, simulation and experiments.

Overall, I find the present study interesting and beneficial to the optics community.

In my view, the manuscript can be accepted for the publication, provided the following concerns are addressed by the authors.

Major Concern:

1. As long as the fiber platform is the GRIN fiber, the MM-solitons are formed comparatively easily. When the authors use a step-index fiber, how they have confirmed that the dispersive radiations are generated from the MM-soliton entities? As far as I know, the group-speeds of the individual spatial modes in a step-index fiber are quite far apart from one another. This may hinder the formation of MM-solitons. In the case of the step-index fibers, the another possibility therefore is that the solitons in the individual modes emit their independent dispersive radiations without locking into a MM-soliton entity.
It would be nice if the authors may shed some light on this.

2. From figure 5 of the main manuscript, it is clear that the authors have used peak power levels of as high as 1.5 MW. For such high peak power levels, I believe that the authors should indicate the approximate values of the silica optical damage threshold as well as the peak power for the catastrophic self-focusing. This would be helpful to the general reader.

Minor Concerns:

1. Line 108: The sizes of the primes over β's is too small to be visible.

2. Line 117: Is a subscript "s" missing below β in the LHS of the equation? I believe that this wavenumber is of the MM soliton constituent in the mode (p,q) and not of the linear dispersive waves. So in my opinion, the authors need to use a subscript "s" in order to distinguish it from the wave number of linear radiation .

3. Line 132: Equation is not in easily readable format. The authors should use a more suitable formatting of the equation.

4. I believe that the authors should provide the form of the gUPPE in the revised manuscript? From the present manuscript, it is not clear whether it is a system of coupled equations like coupled multimode (1+1)-D GNLSEs or a single equation similar to (3+1)-D GNLSE?

5. Line 54-56 : The authors state that "While this theoretical model could elucidate some of the trends in the DW spectrum, it could not account for many of the observed spectral features (Fig. 3 of Ref. [18])."
It would be nice if the authors add two or three sentences to explain what are the specific spectral features of Fig. 3 of ref [18] which can not be explained by using a model encompassing self-imaging (periodic oscillation) behavior of the spatial width of the MM soliton ?

Provided the above concerns are addressed, I recommend the publication of the manuscript.

Editors Comment:
Please provide comments on how the present study relates to
Eftekhar, M.A., Sanjabi-Eznaveh, Z., Lopez-Aviles, H.E. et al. Accelerated nonlinear interactions in graded-index multimode fibers. Nat Commun 10, 1638 (2019). https://doi.org/10.1038/s41467-019-09687-9

Response to Reviewers

First, we would like to thank our reviewers for carefully reading our work and for providing useful suggestions and comments. In revising our paper, we did our best to address our reviewers' concerns. In what follows, we outline the changes made in the order the respective comments appeared in our reviews.

Reviewer #1:

We would like to thank our reviewer for finding our work interesting. In what follows we address the concerns/comments of our reviewer.

- *This very interesting work is quite similar than the one published by the same group in the paper: "Wright, L. G., Wabnitz, S., Christodoulides, D. N. & Wise, F. W. Ultrabroadband Dispersive Radiation by Spatiotemporal Oscillation of Multimode Waves. Phys. Rev. Lett. 115, 223902 (2015)." and referenced as [18] in the paper.*

- While the topic considered in our paper, has commonalities with the work published in Ref. [18], the approach we follow here to explain these effects is radically different. As indicated in our manuscript, the spectral location of the DW spectral lines cannot be explained through the use of Kelly sidebands – a formalism best suited for single mode fibers when they are periodically amplified. What is crucial in determining the position of the DW lines, is the temporal locking of the multimode solitons involved, an aspect that totally escaped previous investigations. Even though, the study of Ref. [18] made a first attempt in investigating these processes, it still fell short in accurately describing and predicting the Cherenkov spectral lines. In particular, in Ref. [18], the mechanism behind DW generation was attributed to the periodic breathing of the pulsed beam as induced in a parabolic-index MMF, a process that is supposed to take place in a gradual manner. However, this is not what we see in our experiments and numerical studies. In fact, the DWs are generated at the point a MM-soliton fission occurs and the DW generation ceases after these events. Moreover, the theory proposed in Ref. [18] cannot explain the generation of DWs in step-index MMFs given that there is no periodic oscillation in these fibers. Even more importantly, the theory presented in our paper, accurately predicts, for the first time, the spectral feature of the DWs generated in multimode optical fibers through the process of multimode soliton locking.

- *Unfortunately this additional work does not provide huge experimental novelty on dispersive wave generation in multimode fibers. As mentioned by the authors, the developed theory of the paper could elucidate some of the trends in the DW spectrum already published in ref [18].*

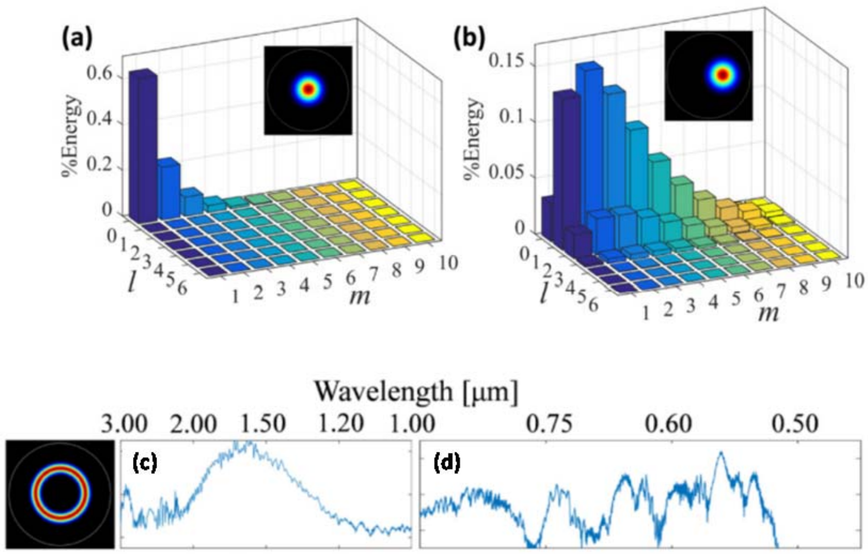
- With all due respect, we beg to disagree with our reviewer. Our work presents novel results in terms of understanding and predicting these very complex effects in nonlinear multimode fiber systems. Nowhere in the literature have so far, these issues been rigorously addressed. These aspects have been discussed in detail in the paragraph above. In addition, what we actually stated in the introduction of our paper, is that Ref. [18] made an attempt to elucidate these phenomena, that statement was not meant for our work.

- *The theory developed in this paper is also not a new and unpublished approach and is mainly based one the work provided by Crosignani and Di Porto (Ref 37 and 38). In these conditions the paper remains interesting to understand more the DW generation in complex multimode fibers.*

- The work by Crosignani and Di Porto was used as a stepping-stone in order to explain DW generation in these multimode systems. We also included some of these elements in our paper for completeness and to facilitate the readership. Nevertheless, nowhere in our paper, we claimed these results. However, the recognition that a massive soliton time-locking effect (via frequency shifts) is actually responsible for these DW combs is new and its experimental validation is reported, for the first time, in our paper. These new results are reflected in Eq. (7) and figures (2)-(5) in our manuscript.

- *Furthermore the authors make statements which should be demonstrated and verified experimentally.*
  Line 207, the authors claim that: "According to Eq. (7), to populate the far reaching DW lines, the beam at the soliton fission point should involve a significant portion of the total energy in higher-order modes", Can the authors provide experimental proof for this claim? Which modes are involved and how many? What means "significant portion"?

- Our simulations indicate that under the excitation conditions used in our experiments (input spot-size) significant energy can be launched in higher-order modes LP_{lm} besides that of the fundamental $LP_{01.}$ A typical launching profile of the modes excited, is shown in figures (a) and (b) below, for both on-axis and off-axis excitation conditions. Of course, once the energy is launched in the MMF, power then migrates via four-wave mixing to other modal groups besides those belonging to LP_{0m}. While, at this point, we are not in position to experimentally demultiplex modes in space (which is in general a difficult task), we can nevertheless discern from the produced DW spectral lines that indeed, higher-order modes (up to LP_{06}) are involved in order to account for the DW lines around 580 THz (see Supplementary material S2). From our previous experience, we also found this to be true (i.e. higher-order modes must be involved), if dispersive wave lines are supposed to appear in the "blue" spectral region (see figures (c) and (d) below). To address these aspects, we have modified lines 209-213 in our paper. We thank our reviewer for bringing this issue up and for helping us improve the clarity of our paper.



- *Can the authors give the excited modes of the fiber during the coupling of the energy and measure the spatial content of the solitons? This would provide an irrefutable proof of the constitution of multimodal solitons and would allow better positioning of theoretical and numerical approaches. Can the author provide experimental proof of the modal constitution of dispersive waves in normal and abnormal dispersion regimes?*

- As we indicated above, at this point, we do not have the capabilities to resolve the output in terms of spatial modes and at different wavelengths. The mode composition can be indirectly inferred in our experiments from the generated spectrum and near-field measurements. This is in accord with our simulations and observations. In addition, mode resolving was not the goal of our studies. We know that several modes are indeed excited and eventually populated through phase-matched processes as evident from the DW lines produced. We hope to address these aspects in future studies when this is technically feasible.

We thank our reviewer for his/her constructive comments.

Reviewer #2:

We would like to thank our reviewer for finding our work comprehensive, interesting and for recommending publication in Communications Physics. Moreover, we thank our referee for bringing up specific issues that helped us improve the clarity of our work.

- *From my understanding, the analytical part of the work is a rather important result and I see only very little discussion on that. I would urge the authors to discuss a bit more in detail the analytical solution. Is that Eq.(1) a derived solution pertaining to Eq.(2) or just an assumed standard soliton solution?, If derived, what technique was used to derive it? As there are multiple methods available to arrive at analytical solutions to complex nonlinear equations like Eq.(2).*

- In equation (1), we express the electric field as a superposition of several propagating modes. After substituting this representation in the nonlinear electrodynamic equations, Eq. (2) follows. The analytical solution given by Eq. (3) is obtained using a standard hyperbolic secant ansatz. This is the only route to obtain a composite soliton solution given that Eq. (2) is not fully integrable and hence cannot be solved using inverse scattering or Hirota methods. Nevertheless, the solution of Eq. (3) does capture the physical attributes observed in our experiments. These issues are now clarified in the revised manuscript lines 100-102.

- *I have noticed that polarization characteristics are underestimated, I appreciate it if the authors could comment on it.*
- In our measurements we integrate both polarizations. This is because small twists in the fiber can induce depolarization. However, this is not an issue since what matters in the work presented here are phase-matched processes that still manifest themselves for each polarization component. We have now clarified this aspect in lines 271-272 in our revised paper.

We thank our reviewer for his/her positive comments and suggestions that have helped us to improve the quality of our work.

Reviewer #3:

We would like to thank our reviewer for finding our work beneficial to the community and for recommending it for publication in Communications Physics. In what follows, we will address our referee's concerns/comments in the order that they appeared in our review.

Major Concern:

- *As long as the fiber platform is the GRIN fiber, the MM-solitons are formed comparatively easily. When the authors use a step-index fiber, how they have confirmed that the dispersive radiations are generated from the MM-soliton entities? As far as I know, the group-speeds of the individual spatial modes in a step-index fiber are quite far apart from one another. This may hinder the formation of MM-solitons. In the case of the step-index fibers, the another possibility therefore is that the solitons in the individual modes emit their independent dispersive radiations without locking into a MM-soliton entity. It would be nice if the authors may shed some light on this.*

- As our reviewer correctly pointed out, the formation of multimode solitons in parabolic index MMFs can readily take place, given that the difference in group velocities is rather small. This explains why MM-soliton formation can observed at a relatively low power levels. Note, that for parabolic fibers, the multimode soliton fission event takes place approximately within the first 10-15 cm in the fiber. Similarly, our simulations carried out in step-index fibers also suggest a similar scenario: the MM-soliton fission occurs within approximately the first 20-30 cm of propagation. The step-index fiber used in our experiments had a numerical aperture of 0.275, a diameter of 105 μm and hence at 1.55 μm, it supported ~1750 modes. Under these conditions, the maximum walk-off time delay between the lowest and the highest-order modes is ~ 25 ps for a length of 30 cm or so. In our experiments, we

estimate that only first 10-20 modes are involved which approximately represent 1% of the total mode population. In this case, the maximum walk-off time between the first mode and 20th mode is expected to be approximately 250-300 fs which is 2-3 times bigger than the pulse-width used in our experiments. Even though these values are higher than those expected from parabolic fibers, the multimode soliton locking still occurs because we used 700 kW that is more than enough to lock the modes by frequency shifting the participating modal components. This information is presented in the section S3 of the supplementary. However, we agree with our reviewer that a MM-soliton, once formed and after its first fission events, will not last long because of walk-off effects. Also, given that the fundamental mode receives a good part of the energy when a fiber is excited with a Gaussian beam profile, we don't believe there is enough amount of energy residing in higher-order modes to form independent single mode solitons.
These aspects are now clarified in section S8 of the Supplementary Material.

- *From figure 5 of the main manuscript, it is clear that the authors have used peak power levels of as high as 1.5 MW. For such high peak power levels, I believe that the authors should indicate the approximate values of the silica optical damage threshold as well as the peak power for the catastrophic self-focusing. This would be helpful to the general reader.*

- The power required at 1.55 μm for catastrophic self-focusing in silica fibers is estimated to be approximately 10 MW. This is roughly 6 times higher than the highest peak powers used in our experiment. We did not notice any damage on the front facet of the large area multimode fibers used and therefore, we cannot provide a good estimate as to the damage threshold (because of ionization) under the pulsed conditions used. These issues are now highlighted in our paper. Please see the bottom 3 lines of the caption for Fig. 5 of the revised manuscript.

Minor Concerns:

1. Line 108: The sizes of the primes over β's is too small to be visible.

This aspect was addressed in the main text.

$$V^{-1} = \beta'_{pq}\left(\omega_{s_{pq}}\right) = \beta'_{pq}\left(\omega_0\right) + \beta^{(2)}_{pq}\left(\omega_0\right)\Omega_{pq}$$

2. Line 117: Is a subscript "s" missing below β in the LHS of the equation? I believe that this wavenumber is of the MM soliton constituent in the mode (p,q) and not of the linear dispersive waves. So in my opinion, the authors need to use a subscript "s" in order to distinguish it from the wave number of linear radiation.

We thank our reviewer for pointing out this typographical error.

$$\beta_{s_{pq}}\left(\omega\right) = \beta_{s_{pq}}\left(\omega_{s_{pq}}\right) + \frac{1}{V}\left(\omega - \omega_{s_{pq}}\right) + \frac{\left|\beta^{(2)}_{pq}\right|}{2\tau^2}, \quad (5)$$

3. Line 132: Equation is not in easily readable format. The authors should use a more suitable formatting of the equation.

We changed the format of the equation. By removing some extra parentheses and changing the sizes of the others, I hope we could make the equation easier to read.

$$\beta_{mn}(\omega) = k_0 n(\omega)\left[1 - \left(2\sqrt{2\Delta}(m+n+1)/k_0 a n(\omega)\right)\right]^{1/2}$$

4. I believe that the authors should provide the form of the gUPPE in the revised manuscript? From the present manuscript, it is not clear whether it is a system of coupled equations like coupled multimode (1+1)-D GNLSEs or a single equation similar to (3+1)-D GNLSE?

A description of the gUPPE code used is now provided in the supplementary section S9.

5. Line 54-56 : The authors state that "While this theoretical model could elucidate some of the trends in the DW spectrum, it could not account for many of the observed spectral features (Fig. 3 of Ref. [18])." It would be nice if the authors add two or three sentences to explain what are the specific spectral features of Fig. 3 of ref [18] which can not be explained by using a model encompassing self-imaging (periodic oscillation) behavior of the spatial width of the MM soliton ?

We have highlighted these specific aspects concerning the spectral lines in our revised manuscript. Please see line 56 of the revised manuscript.

We would like to thank our reviewer for his/her helpful comments and suggestions. They truly helped us to improve the clarity of our work.

REVIEWERS' COMMENTS:

Reviewer #1 (Remarks to the Author):

The answers given by the authors seem good to me

Reviewer #2 (Remarks to the Author):

After having reviewed the revised version of the manuscript , I see the authors put significant effort to rework the manuscript taking into the suggestion made by the reviewers during the first round of revision.
I am also happy with the detailed response to my questions. I am convinced with the present form of the manuscript, and it is my considered opinion that the article can be accepted for publication.

Reviewer #3 (Remarks to the Author):

I am satisfied with the authors response and recommend the publication of the manuscript.