Robustness of Noisy Quantum Networks - Supplementary Information

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SUPPLEMENTARY NOTES

Supplementary Note 1. RESOURCES CONSUMPTION FOR A SIMPLE QUANTUM REPEATER PROTOCOL

We consider quantum repeater protocols with a resource consumption per link that grows as $r(l) = l^{\alpha}$. An exact evaluation of α is impossible since it depends heavily on the experimental apparatus used for the repeater scheme and the noise present in it. To provide an intuition for the possible values of α in this section we compute $r(l)$ for an idealized quantum repeater protocol. For simplicity, we consider average resource consumption and the predicted fidelity generated with that amount of entangled qubits. Our methodology allows us to also take into account a model when a certain fidelity is generated with a certain probability, but give that in this section we are only interested in the resource consumption scaling, we will use averages instead for simplicity. Let us consider a nested protocol in a quantum repeater chain [1, 2] which is presented in Fig. S3. Let us consider the chain in Fig. S3, where nodes are connected through entangled qubits with fidelity f_{in}. In a nested protocol entangled qubits are purified in rounds. In each round two entangled qubits are purified to obtain an entangled qubit with higher fidelity, and in turn, those entangled qubits are used in the next round to, once again, increase the fidelity of the entangled qubit, until a fidelity f_2 is reached (Fig. S3 (b)). After this, swapping is applied in some vertices to create entanglement between nodes at distance two (or more) from each other (Fig. S3 (c)). the predicted fidelity of the entangled qubits drops to a fidelity f_3 during this last step, but given that the entangled qubit were purified beforehand, it is possible keep the predicted fidelity above a certain threshold fidelity. This creates a single link with entanglement purification followed by a swapping protocol is then used again to create entanglement between nodes at distance 4 (or more) from each other, and the procedure is repeated until there is one entangled qubit between the two endpoints of the chain.

The goal is to use the right number of rounds of purification so that the predicted fidelity between the endpoints of the chain is the same as the initial fidelity. Figure S2 (a) shows the fidelity of a Werner state, with initial fidelity 0.97 that was purified using s rounds of purification and then swapped m times (m is equal to the network distance minus 1). Notice that when considering one round of purification the fidelity drops below the threshold 0.99, after only one swap. Therefore, for this setup, at least two rounds of purification are necessary keep the predicted fidelity from dropping below the initial fidelity 0.99. Let us now calculate how many entangled qubits will be consumed in order to create a link between two nodes at distance l. We consider the following protocol. Entangled qubits are purified s times, then swapped until right before the fidelity drops bellow 0.99, and then the entangled qubits are purified again s times. Because entanglement purification is a stochastic process, we will consider the average number of entangled qubits consumed, given that for each round of purification the number of entangled qubits consume in average is equal to 2 divided by the probability of the purification process being successful. Figure S2 (b) shows the average number of entangled qubits consumed, as function of the distance between nodes. We obtain a scaling with α somewhere between 1 and 2. More inefficient protocols can yield better results, but imperfect quantum gates

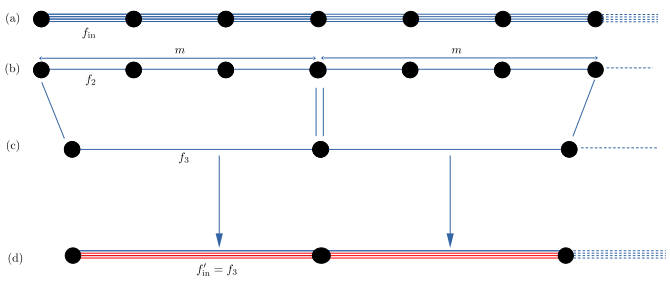

Figure S1. **Illustration of a nested quantum repeater protocol** Depicted in (a) is a quantum repeater with q entangled qubits pairs shared between stations with a fidelity f_{in}, in the illustration we consider $q = 4$. Those entangled qubits pairs can be purified to create a single entangled qubit pair shared between stations, with a higher fidelity f_3, as shown in (b). After this, swapping is applied in m intermediary station, to create entangled qubits between stations at distance m with fidelity f_3, shown in (c). The goal is to select the largest m so that $f_3 > f_{\text{targer}}$. This process creates a single entangled qubit pair between stations at distance $m + 1$, as shown in (d). To then repeat this process and create entangled qubit pair between stations at distance $2m$, we will need q of this entangled qubit pair with fidelity f_3 that can be seen as our new starting fidelity f'_{in}, as depicted in (d). This process is then repeated until the entangled qubits reach the endpoints of the chain.

but quantum memory times will hinder the efficiency of the quantum repeater protocol. The exact value of α will only be known after the development of quantum repeaters in laboratories, and will likely change over time.

Supplementary Note 2. RELATION BETWEEN THE FUNCTIONALLY CONNECTED COMPONENT AND THE MAXIMUM CLIQUE

The largest functionally connected component defined in the main text is the largest set of nodes such that for any two nodes can individually establish connections with sufficient fidelity with each other. A clique is a subset of nodes such that there is a link between them. If a problem is at least as hard as an NP-hard, then the problem is also NP-hard, If for all pairs of ij, n_{ij} is either 0 or 1, then our problem is equivalent to solving the maximum clique problem that is a known NP-hard problem [3]. Therefore we can conclude that finding the largest functionally connected component is at least as hard as solving the maximum clique problem making this problem NP-hard.

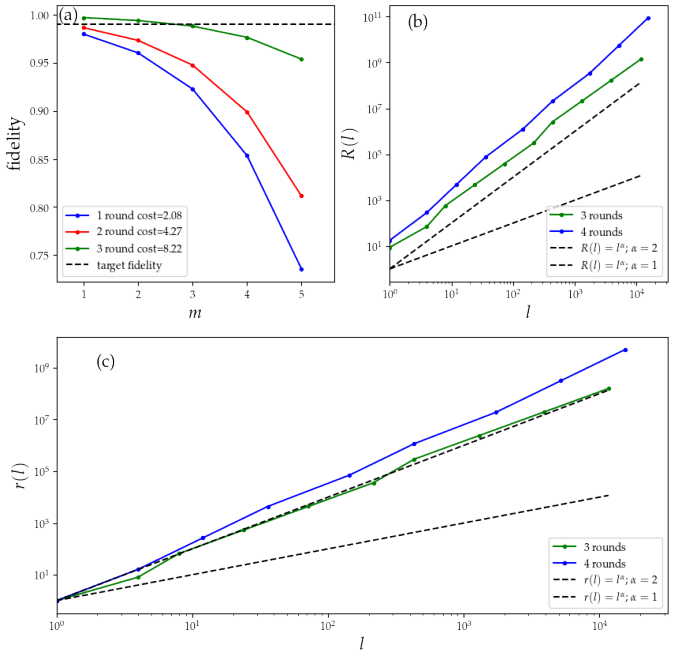

Figure S2. **Resources consumption of a toy nested repeater protocol**. Depicted in (a) is the Fidelity of a Werner state after 1,2,3 rounds of purification followed by $m-1$ swappings is shown in (a). We start with a Werner state with fidelity 0.97, and an equal target fidelity. (b) Shows the number of entangled qubits consumed $R(l)$ as a function of l for a simple nested protocol with 2 and 3 rounds of purification. For this protocol $r(l)$ scales roughly between l^2 and l. $r(l)$ is defined as the number of entangled qubits consumed to create a link with the target fidelity at distance l dived by the number of entangled qubits necessary to create one entangled qubits with the target fidelity at distance 1. (b) Shows the $r(l)$ a function of l for a simple nested protocol with 2 and 3 rounds of purification. For this protocol $r(l)$ scales roughly between l^2 and l.

Supplementary Note 3. THE ERDŐS-RÉNYI NETWORK

Here we will calculate the operational an dimater fnction $l^{\text{op}}(p^{\text{op}})$ and $D(p^{\text{op}}p^{\text{ext}})$ analytically for for large networks $N \gg 1$ and an exponential distribution of entangled qubits across the links. Consider an Erdős-Rényi network with N and an average degree c. The diameter of an Erdős-Rényi networks in the supercritical regime ($c > 1$) is given by [4]

$$d/\ln N = \frac{1}{\ln c} - \frac{2}{\ln c^*} + \frac{O(1)}{\ln N} \tag{S1}$$

where c^* is the solution of $c^* \exp(-c^*) = c \exp(-c)$ constrained by $0 < c^* \leq 1$. For $\ln N \gg 1$ the last term can be discarded. Next in the subcritical regime $c < 1$, far enough from the critical points, the diameter is given by

$$\frac{d}{\ln N} = -\frac{1}{\ln c} - \frac{2}{\ln c \ln N} - \frac{O(1)}{\ln N} \tag{S2}$$

In the limit $\ln N \gg 1$ we have,

$$d/\ln N = \begin{cases} \frac{1}{\ln c} - \frac{2}{\ln\ c^*} & c > 1 \\ -\frac{1}{\ln c} & c < 1. \end{cases} \tag{S3}$$

where we note that this is scaling is more rigorous than the traditional scaling $\ln N/\ln c$, found in most books [5]. When comparing with simulation we found the extra term to be not negligible. Now let us consider an example of an Erdős-Rényi network with N nodes and an average degree c. Considering that links remain in the network with a total probability of $p = p^{\rm op}p^{\rm ext}$ which is the probability that we have sufficient pairs to create our link.

$$\frac{D(p^{\rm op}p^{\rm ext})}{\ln N} = \begin{cases} \frac{1}{\ln(cp^{\rm op}p^{\rm ext})} - \frac{2}{\ln\ (cp^{\rm op}p^{\rm ext})^*} & cp^{\rm op}p^{\rm ext} > 1 \\ -\frac{1}{(cp^{\rm op}p^{\rm ext})} & cp^{\rm op}p^{\rm ext} < 1. \end{cases} \tag{S4}$$

For convenience let us define two new variable $x \equiv -\ln p^{\rm op}$ and $y \equiv -\ln p^{\rm ext}$, so we can write,

$$\frac{D(x+y)}{\ln N} = \begin{cases} \frac{1}{\ln c-(x+y)} - \frac{2}{\ln\left(c\ e^{-(x+y)}\right)^*} & c\ e^{-(x+y)} > 1 \\ -\frac{1}{\ln c-(x+y)} & c\ e^{-(x+y)} < 1. \end{cases} \tag{S5}$$

Considering, that the number of entangled pairs in each link follow an exponential distribution with mean number $\langle n\rangle$ (which is our distribution used in the main text), the probability that a link contain more than $(l^{\rm op})^\alpha$ entangled qubit pairs is given by $p((l^{\rm op})^\alpha) = \exp\left(-(l^{\rm op})^\alpha/\langle n\rangle\right)$. In turn this means,

$$l^{op}(x) = x^{1/\alpha}\langle n\rangle^{1/\alpha}. \tag{S6}$$

Where l^{op} is the operation distance written a function of $x \equiv -\ln(p_{\rm fixed})$. Combining eqn (S6) with with eqn (S5), we obtain

$$x_0^{1/\alpha}\frac{\langle n\rangle^{1/\alpha}}{\ln N} = \begin{cases} \frac{1}{\ln c-(x_0+y)} - \frac{2}{\ln\left(c\ e^{-(x_0+y)}\right)^*} & c\ e^{-(x_0+y)} > 1 \\ -\frac{1}{\ln c-(x_0+y)} & c\ e^{-(x_0+y)} < 1. \end{cases} \tag{S7}$$

From here we can derive that to maintain the proprieties of the network constant, for large number of nodes N, the average number of qubits should grow as $\langle n\rangle \sim (\ln N)^\alpha$. Solving eqn. (S7) yields the same behavior displayed in Fig. 2, with the difference that the solution in the subcritical regime is always present, as shown in Fig. S3.

If we consider a different distribution of entangled qubits the results will change qualitatively but not quantitatively. For an uniform distribution the probability of having n entangled qubits in an edge is given by,

$$g(n) = \begin{cases} \frac{1}{n_{\rm max}-n_{\rm min}} & n_{\rm min} < n < n_{\rm max} \\ 0 & \text{otherwise} \end{cases} \tag{S8}$$

where $n_{\rm min}$ ($n_{\rm max}$) is the minimum (maximum) number of entangled qubits that a link can have. Setting $n_{\rm min} = 0$, ($n_{\rm max}$) is then related to the mean number of entangled qubits by $n_{\rm max} = 2\langle n\rangle$. Using eqn (1) from the main paper with this distribution yields,

$$p_{\rm fixed}(l_{\rm fixed}) = \begin{cases} 1 - \frac{(l_{\rm fixed})^\alpha}{2\langle n\rangle} & l_{\rm fixed} < (n_{\rm max})^{2\langle n\rangle} \\ 0 & \text{otherwise} \end{cases} \tag{S9}$$

$l^{\rm op}$ as a function of $x \equiv p^{\rm op}(l^{\rm op})$ can be written as,

$$l^{\rm op}(x) = \left[2\langle n\rangle\left(1-e^{-x}\right)\right]^{1/\alpha}. \tag{S10}$$

Combining eqn (S6) with eqn (S7), one obtains,

$$\left(1-e^{-x_0}\right)^{1/\alpha}\frac{\langle n\rangle^{1/\alpha}}{\ln N} = \begin{cases} \frac{1}{\ln c-(x_0+y)} - \frac{2}{\ln\ c^*} & c\ e^{-(x_0+y)} > 1 \\ -\frac{1}{\ln c-\tilde{x}_0} & c\ e^{-(x_0+y)} < 1. \end{cases} \tag{S11}$$

This gives the same resource scaling and similar results has we have previously seen, as shown in Fig. S3.

Supplementary Note 4. CONDITIONS FOR THE SUPPRESSION OF THE FIRST ORDER PHASE-TRANSITION

We will be using the variable defined in supplementary Note 3, For an exponential distribution of entangled qubit pairs, eqn (3) can be expressed as,

$$(\tilde{x}-y)\langle n\rangle = D(\tilde{x})^\alpha \tag{S12}$$

where we do one more change of variables $\tilde{x} = x + y$. If for each value of y there is only solution for eqn (S12) then this phase-transition will be continuous. To check this we can express y as a function of x which is given

$$y(\tilde{x}) = \tilde{x} - \frac{D(\tilde{x})^\alpha}{\langle n\rangle} \tag{S13}$$

Given a value of y, $\tilde{x}_0 \equiv x_0 + y$ can be found from $y(\tilde{x}_0) = y$. If $y(\tilde{x})$ is a monotonically decreasing function for $0 < \tilde{x} < y_{c_3}$, then the transition is continuous, where $y_{c_3} = -\ln p_c$. The region for $\tilde{x} \geq y_{c_3}$ is not important since at this point the network is disconnected. In contrast, if the function is not a monotonically decreasing function,

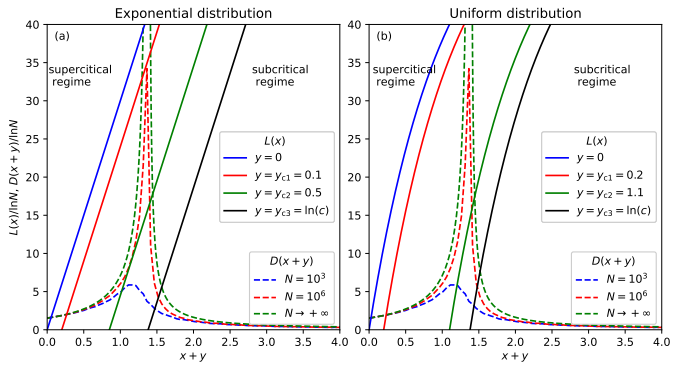

Figure S3. **Asymptomatic robustness of a quantum Erdős-Rényi network**. Exploration of bond percolation for an Erdős-Rényi network with N nodes and an average degree $c = 4$ where the number of entangled pairs in each link follows an exponential distribution with mean number $\langle n\rangle$ (a). and here the number of entangled pairs in each link follow an uniform distribution with mean number $\langle n\rangle$, (b). We plot $l^{\rm op}(x)$ and $D(x+y)$ versus $x+y$ for $\langle n\rangle = (30\ln N)^{\alpha}$ with $\alpha = 1$. Theoretical prediction for $D(x+y)$ when $N \to +\infty$ shown as a green dashed curve. $y_{\rm c1(c2)}$ gives the smallest(largest) point where the networks can break apart functionally while $y_{\rm c3}$ gives the point the network is not in the functionally supercritcal regime.

then one can say that in general the phase transition will not be continuous. For $\alpha \neq 0$ the phase transition is only continuous if

$$\langle n\rangle \geq n_{\rm c}, \tag{S14}$$

where n_c is defined as,

$$n_{\rm c} = \alpha \max_{0<\tilde{x}<y_{c_3}} \left[D(\tilde{x})^{\alpha-1} \frac{d}{d\tilde{x}} D(\tilde{x}) \right]. \tag{S15}$$

Furthermore, there is a value α_c such that for $\alpha < \alpha_c$, there will be no discontinuous phase transition for any value of $\langle n\rangle$. To derive such a value one needs to consider that the network is in the supercritcal regime when no links are removed from the network, $y = 0$. For a given value of y and $\tilde{x}$, the mean number of entangled qubits in the network is,

$$\langle n\rangle = \frac{D(\tilde{x})^{\alpha}}{\tilde{x} - y}. \tag{S16}$$

Due to the fact that $y \geq 0$, one has that the mean number entangled qubits in the network must be greater than the critical value

$$\langle n\rangle^{*} = D(\tilde{x})^{\alpha}/\tilde{x}. \tag{S17}$$

for any value of $\tilde{x}$ in the range $x < y_{c_3}$. This mean number entangled qubits $\langle n\rangle$ for the network to be in the connected regime when no links are removed from the network. For a network with n_c larger or equal to $\langle n\rangle^{*}$, in the range $\tilde{x} < y_{c_3}$, there can be no discontinuous phase transition. Adding this constraint to eqn (S15), one obtains that, independently of the amount of resource in the network, the discontinuous phase transition is suppress if $\alpha < \alpha_c$, with α_c given by,

$$\alpha_c = \min_{x<y_{c_3}} \frac{D(x)/x}{dD(x)/dx}. \tag{S18}$$

Supplementary Note 5. RELATION BETWEEN p_c AND $p_{c2}^{\rm ext}$

We will be using the variable defined in supplementary Note 3. Let us now establish the relationship between two important quantities, $y_{\rm c1} = -\ln p_{c2}^{\rm ext}$ gives the point where the networks breaks functionally while $y_{\rm c3} = -\ln p_c$ gives the point the network is not in the functionally supercritcal regime. Considering that our resources are exponentially distributed, the relation between $y_{\rm c1}$ and $y_{\rm c3}$ can be computed using eqn (S12). It is given by

$$y_{\rm c1} = y_{\rm c3} - \frac{D(y_{\rm c3})^{\alpha}}{\langle n\rangle} \tag{S19}$$

Using the fact the diameter of the network diverges at $y_{\rm c3}$ meaning $D_{\rm max} = D(y_{\rm c3})$, one obtains

$$y_{\rm c1} = y_{\rm c3} - \frac{(D_{\rm max})^{\alpha}}{\langle n\rangle} \tag{S20}$$

which is equivalent to eqn (5) in the main paper.

Supplementary Note 6. SCALE OF GEOMETRIC GRAPHS

In this section we explain how the scale for the geometric networks was calculated. In a random geometric graph, N nodes are randomly distributed in a d-dimensional space of length L. Two nodes are connected to each other by a link if the distance between them is less than R. For a two-dimensional space, the average degree of the network, c relates to N, R and L by [6]

$$c = \frac{N\pi R^2}{L^2} \tag{S21}$$

Note that in [6] L is fixed to be one, but one can easily re-scale R as a function of L to arrive at eqn (S21). In our work, we consider that the density of nodes per km is constant and therefore L increases with N as

$$L = \sqrt{\frac{\pi}{c}} R\sqrt{N}, \tag{S22}$$

and the relation between L and $\sqrt{N}$ depends on R and c. We consider an average degree of c equal to 8 and 10 in our network. This values are large enough to guarantee that the networks are in the connected but small that we

	R=20 km			R=266 km		
	c=8	c=10	Order of	c=8	c=10	Order of
$N = 10^{2.5}$	223	199	$\sim 10^2$ km	2 518	2 253	$\sim 10^3$ km
$N = 10^3$	396	355	$\sim 10^3$ km	4 479	4 006	$\sim 10^4$ km
$N = 10^{3.5}$	705	630		7 964	7 123	
$N = 10^4$	1 253	1 121		14 163	12 667	
$N = 10^{4.5}$	2 228	1 993		25 185	22 526	
$N = 10^5$	3 963	3 545	$\sim 10^4$ km	44 786	40 057	$\sim 10^5$ km

Table S1. Network scale versus the number of nodes for a random geometric network.

can consider the networks spares, even for small values o N. The second variable that we need to fix in order to calculate L as function of N is the connecting distance R. The value of R is not know and it will depend on the technology used. Table (S1) shows the scale L for different value of linking distance, $R = 226$ km (considered in [7]) and a more conservative value of $R = 20$ km. For $R = 226$ km, $N = 10^2.5$ nodes corresponds to distance of the order of 10^3 km, that is more or less the scale of a small country. $N = 10^5$ on the other corresponds to distances of the order of 10^5 km that is comparable to the circumference of earth (40 075 km). So for a linking distance of $R = 226$ km or work consists of a scale that varies from the size of a small country to values on the order the entire earth. For $R = 20$ km all the values are one order of magnitude lower.

Supplementary Note 7. WAXMAN MODEL SCALE

In the Waxman model the probability that two nodes are connect decays with the distance as [8]

$$p_{ij} = \beta e^{-\frac{r_{ij}}{R}} \tag{S23}$$

with r_{ij} being the distance between node i and j, and R the average connection distance, and in our work we consider $\beta = 1$. If one considers that N nodes are randomly distributed in over a two dimensional space of length L, and $R \ll L$ one can do a quick estimate the average degree of the network using a similar approach to the one introduced in [6]. The probability that a node is connected to a node at distance r is given the the probability that a node exists at distance r, $(2N\pi r\partial r/L^2)$, times the probability that a link at that distance is successfully generate, given by eqn (S23). Integrating over all value of r, one obtains

$$c = \frac{N}{L^2}\int_0^\infty e^{-\frac{r}{R}} 2\pi r \partial r = \frac{2N\pi R^2}{L^2}. \tag{S24}$$

In our work we consider that density of nodes per km is constant and therefore L increases with N as

$$L = \sqrt{\frac{2\pi}{c}} R\sqrt{N}. \tag{S25}$$

Given this preliminary estimation we can conclude the for a a fix number or nodes L and degrees c and connecting radios R the scale of the Waxman model is $\sqrt{2}$ times bigger scale of the random geometric graph, but still in the same order of magnitude.

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