Ultimate Speed Limits to the Growth of Operator Complexity —Supplementary Information—

Niklas Hörnedal[1,2], Nicoletta Carabba[1], Apollonas S. Matsoukas-Roubeas[1], and Adolfo del Campo[1,3]

[1]Department of Physics and Materials Science, University of Luxembourg, L-1511 Luxembourg, G. D. Luxembourg
[2]Fysikum, Stockholms Universitet, 106 91 Stockholm, Sweden
[3]Donostia International Physics Center, E-20018 San Sebastián, Spain

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Supplementary note 1: Explicit models

In this appendix we introduce three dynamical models that, having the structure of a closed complexity algebra, display a maximal growth of complexity, in the sense that the complexity rate saturates the dispersion bound. Moreover, we show that quantum chaos, in the Hamiltonian sense, is not necessary to have maximal complexity growth. In particular, it is shown that the dynamics of simple solvable Hamiltonians can saturate our bound.

We first consider a finite-dimensional model, namely the SU(2) algebra, and then turn to the infinite-dimensional case, which allows us to comment on the famous conjecture by Parker *et al.* [1]. In particular, we show that our notion of maximal complexity growth is more general than the one proposed in their work, as the latter represents a special case of the former.

SU(2) algebra

Let us start with the SU(2) algebra $[J_i, J_j] = i\epsilon_{ijk}J_k$. That is, let us consider the dynamical evolution generated by the Liouvillian

$$\mathcal{L} = \nu(J_+ + J_-), \tag{1}$$

where $J_\pm = J_1 \pm iJ_2$ are the familiar SU(2) ladder operators and $\mathcal{L}_\pm = \nu J_\pm$. Now, the Krylov basis corresponds to the usual basis of the representation j: $|\mathcal{O}_n) = |j, n\rangle$ with $-j \leq n \leq j$. Following [2], let us relabel the vectors with $n \to n + j$, so that $n = 0, \dots, 2j$, the dimension of the Krylov space being equal to $2j + 1$. By construction, the initial operator $|\mathcal{O}_0)$ is just the highest weight state $|j, -j\rangle$ and is annihilated by J_-. From the action of the ladder operators on the representation basis:

$$J_+ |j, -j+n\rangle = \sqrt{(n+1)(2j-n)}\, |j, -j+n+1\rangle\,, \tag{2}$$
$$J_- |j, -j+n\rangle = \sqrt{n(2j-n+1)}\, |j, -j+n-1\rangle\,, \tag{3}$$

being $\mathcal{L}_-|\mathcal{O}_n) = b_n|\mathcal{O}_{n-1})$, we can read off the Lanczos coefficients:

$$b_n = \nu\sqrt{n(2j+1-n)}. \tag{4}$$

The Heisenberg evolution of an operator can be understood as the displacement of a generalized coherent state [2]:

$$|\mathcal{O}(t)) = e^{i\mathcal{L}t}|\mathcal{O}_0) = D(\xi = i\nu t)|\mathcal{O}_0). \tag{5}$$

Indeed, the displacement operator is defined as

$$D(\xi) = e^{\xi J_+ - \xi^* J_-}. \tag{6}$$

The generalized coherent state $|\xi, j\rangle$ can be expanded over the spin basis $|j, -j+n\rangle$ as follows [2]:

$$|\xi, j\rangle = (1 + |\xi|^2)^{-j} \sum_{n=0}^{2j} \xi^n \sqrt{\frac{\Gamma(2j+1)}{n!\Gamma(2j-n+1)}} \, |j, -j+n\rangle \, . \tag{7}$$

For this model it is convenient to use complex polar coordinates $\xi = e^{i\phi} \tan\theta$. Indeed, by replacing $\theta = \nu t$ and $\phi = \pi/2$ and by using the correspondence between the spin and the Krylov basis $|\mathcal{O}_n) = |j, -j+n\rangle$, from above one can read the components of the operator wavefunction:

$$\phi_n(t) = \tan^n(\nu t) \cos^{2j}(\nu t) \sqrt{\frac{\Gamma(2j+1)}{n!\Gamma(2j-n+1)}}, \tag{8}$$

from which we can compute both the mean (i.e. the Krylov complexity) and the variance of the complexity operator $\mathcal{K}$:

$$K(t) = \sum_{n=0}^{2j} n \, \phi_n^2(t) = 2j \sin^2 \nu t, \tag{9}$$

$$\Delta\mathcal{K}(t) = \sqrt{\sum_{n=0}^{2j} n^2 \phi_n^2(t) - K^2} = \sqrt{\frac{j}{2}} |\sin 2\nu t|, \tag{10}$$

Since $b_1 = \nu\sqrt{2j}$, one can check that $|\partial K| = 2 b_1 \Delta\mathcal{K}$ at any time t: that is, as expected from the closure of the 3-dimensional complexity algebra, the dispersion bound is identically saturated.

Given the expression of the Liouvillian in Krylov space, it is generally a difficult task to derive a corresponding Hamiltonian that generates the dynamics in the Hilbert space. In particular, the former contains less information than the latter and therefore many different Hamiltonians can give rise to the same dynamics in Krylov space. Moreover, one has not only to specify the Hamiltonian but also the initial operator $\mathcal{O}_0$. Nevertheless, we find that the evolution of the operator $\mathcal{O}_0 = \sigma_1 + \sigma_3$ under the single-qubit (two-level) Hamiltonian $H = \nu\sigma_3$, where σ_i is the i-th Pauli matrix, is given in Krylov space by the representation $j = 1$ of the SU(2) algebra. More precisely, by explicitly performing the Lanczos algorithm, which in this case involves only two steps, we find Lanczos coefficients $b_1 = b_2 = \nu\sqrt{2}$, which coincide with Eq. (4) for $j = 1$. We note that here the dimension of the Krylov space is $D = 3$, which is the maximum allowed for a Hilbert dimension $d = 2$, being $D \leq d^2 - d + 1$ [3]. This is achieved due to the choice made for the initial operator $\mathcal{O}_0$, which has non-zero components along all the Liouvillian eigenspaces. If instead one starts with an initial operator $\mathcal{O}_0 = \sigma_1$, the Krylov dimension shrinks to $D = 2$, in which case the bound is always trivially saturated, being the complexity algebra given by the representation $j = 1/2$ of SU(2). From this example, we deduce that non-chaotic Hamiltonian can give rise to maximal complexity growth in Krylov space. Interestingly, the same observation was made also in [4] with respect to the different notion of maximal complexity growth proposed by Parker *et al.* [1], proving that the exponential growth of complexity can be achieved also without chaos.

As a final remark, let us note that by considering a more general two-level Hamiltonian $H = c\mathbb{1} + \vec{v} \cdot \vec{\sigma}$ we can still obtain the same dynamics in Krylov space, i.e. representation $j = 1$ of SU(2), provided that we tune the parameters and we choose the initial operator in such a way that $b_1 = b_2$: if this condition does not hold, the bound cannot be saturated. More generally, in any Krylov space of dimension $D = 3$, the algebraic closure and thus the saturation of the bound is possible if and only if $b_1 = b_2$, i.e. if and only if the underlying algebra is given by the representation $j = 1$ of SU(2). This can be checked by explicitly computing the double commutator $[\mathcal{L}, [\tilde{\mathcal{K}}, \mathcal{B}]]$ and observing that it vanishes only if $b_1 = b_2$. Indeed, as emphasized in the main text, the only possible closed complexity algebra in the case of a finite Krylov dimension D is given, up to a multiplicative constant, by the representation $j = \frac{D-1}{2}$ of SU(2).

Heisenberg-Weyl algebra

Let us now consider the case of infinite-dimensional Krylov space. An emblematic example in which the bound is saturated is the one in which the dynamical evolution is given in terms of the Heisenberg-Weyl (HW) algebra $[a, a^\dagger] = 1$. In this case, the Liouvillian is given by

$$\mathcal{L} = \nu(a^\dagger + a), \tag{11}$$

and the generalized ladder operators $\mathcal{L}_\pm$ are just the raising and lowering operators $a^\dagger$ and a, times the constant ν. Here the initial operator $|\mathcal{O}_0)$ is represented as the vacuum state $|0\rangle$ and the Krylov basis corresponds to the usual basis constructed by acting with $a^\dagger$ on the vacuum:

$$|\mathcal{O}_n) = |n\rangle \, \frac{1}{\sqrt{n!}}(a^\dagger)^n \, |0\rangle \,, \tag{12}$$

that is, the eigenbasis of the number operator $a^\dagger a$, which coincides with the complexity operator $\mathcal{K}$. We note that in this case the Krylov space has infinite dimension. From the well known relations

$$a^\dagger \, |n\rangle = \sqrt{n+1} \, |n+1\rangle \,, \quad a \, |n\rangle = \sqrt{n} \, |n-1\rangle \,, \tag{13}$$

one can see that $b_n = \nu\sqrt{n}$. The time-evolved operator $|\mathcal{O}(t))$ can be represented as the standard coherent state

$$|\xi\rangle = D(\xi) \, |0\rangle = e^{-|\xi|^2/2} \sum_{n=0}^{\infty} \frac{\xi^n}{\sqrt{n!}} \, |n\rangle \tag{14}$$

for $\xi = i\nu t$. Therefore, the components of the operator wavefunction are

$$\phi_n(t) = e^{-(\nu t)^2/2} \frac{(\nu t)^n}{\sqrt{n!}}, \tag{15}$$

from which we can compute that

$$K(t) = (\Delta\mathcal{K})^2 = \nu^2 t^2, \tag{16}$$

We thus conclude that, being $b_1 = \nu$, the dispersion bound is always saturated: that is, $|\partial K| = 2b_1\Delta\mathcal{K}$ $\forall t$. This model provides an example in which maximal complexity growth (in the sense of saturation of our bound) is achieved, while the conjecture by Parker *et al.* [1], i.e. linear growth of Lanczos coefficients, does not hold. We therefore see that the two notions of maximal complexity growth are not equivalent.

SYK model

Finally, let us consider the celebrated prototype for quantum chaos: the SYK model of N Majorana fermions with q-body interaction, given by the Hamiltonian

$$H^{(q)}_{SYK} = i^{q/2} \sum_{1 \le i_1 < i_2 < \cdots < i_q \le N} J_{i_1 \ldots i_q} \gamma_{i_1} \ldots \gamma_{i_q}. \tag{17}$$

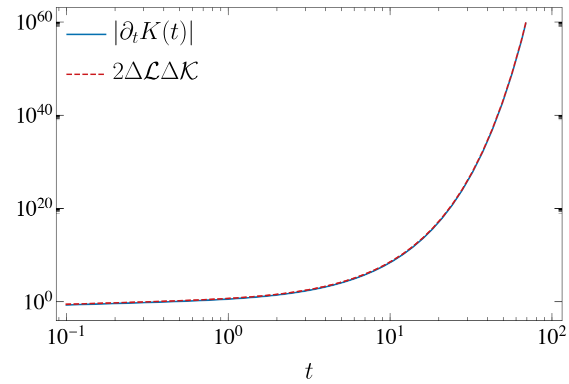
In the large-N limit the model can be solved analytically and, for asymptotically large q, has been proven to obey the universal growth hypothesis by Parker *et al.* [1]: namely, the growth of the Lanczos coefficients is asymptotically linear in n, resulting in the exponential time-behaviour of Krylov complexity. More precisely, it can be shown that, in this limit, the SYK belongs to a family of exact solutions with Lanczos coefficients [1]

$$b_n = \nu\sqrt{n(n-1+\eta)} \tag{18}$$

and amplitudes

$$\phi_n(t) = \sqrt{\frac{(\eta)_n}{n!}} \tanh^n(\nu t) \, \mathrm{sech}^\eta(\nu t), \tag{19}$$

where $(\eta)_n = \eta(\eta+1)\ldots(\eta+n-1)$ is the Pochhammer symbol. From these amplitudes one can extract the complexity $K(t) = \eta \sinh^2(\nu t)$, which, as expected from the asymptotic linear behaviour of the Lanczos coefficients, shows an asymptotic exponential growth. Remarkably, the linear growth of the Lanczos coefficients is a sufficient (but not necessary, as shown above) condition for the saturation of



Supplementary Figure 1 – For the SYK model, the complexity rate (solid light blue line) saturates the dispersion bound (dashed red line) at any time. In the plot we fix the parameters appearing in Eq. (18) to be $\nu = \eta = 1$, that is we consider an exact linear growth of the Lanczos coefficients: $b_n = n$ for $n > 1$.

the dispersion bound on complexity, as shown in Supplementary Figure 1. This saturation is due to the presence of an underlying complexity algebra: indeed, one of the main results of our work is the proof that the closure of the complexity algebra is both a sufficient and a necessary condition for the dispersion bound to be saturated. For this particular family of solutions, the underlying algebra is that of $\mathrm{SL}(2,\mathbb{R})$ [2].

Supplementary note 2: Equivalence between the saturation of the dispersion bound and the simplicity hypothesis

In this appendix, we show that there is an equivalence between the saturation of the dispersion bound and the simplicity hypothesis being satisfied. When we say that the complexity algebra is closed, we will simply mean that the simplicity hypothesis is satisfied.

The right-hand side of the dispersion bound is equal to two times the norm of the vectors $(\mathcal{K} - \langle\mathcal{K}\rangle_t)\mathcal{O}(t)$ and $(\mathcal{L} - \langle\mathcal{L}\rangle_t)\mathcal{O}(t)$, while the left-hand side is obtained by applying the Cauchy-Schwarz inequality. From this, it is clear that the bound is saturated if and only if the two vectors are linearly dependent. In other words, the bound is saturated if and only if the vectors $(\mathcal{K} - K)|\mathcal{O}(t))$ and $\mathcal{L}|\mathcal{O}(t))$ are linearly dependent, where we have chosen to suppress the time dependence of K. What will follow is a series of steps proving that the complexity algebra being closed is both necessary and sufficient for the vectors $(\mathcal{K} - K)|\mathcal{O}(t))$ and $\mathcal{L}|\mathcal{O}(t))$ to be linearly dependent. Said differently, the complexity algebra being closed is equivalent to the dispersion bound being saturated. When carrying out the proofs, we will use the convention that $b_0 = 0$ and for finite Krylov dimension D, we will also introduce $b_D = 0$. For any superoperator M we will write $\mathrm{M}_{n,m} \equiv (\mathcal{O}_n|\mathrm{M}\mathcal{O}_m)$, where $\mathrm{M}_{n,m}$ can be thought of as the entries of a matrix representing M.

Proving necessity

Linear dependence between $(\mathcal{K} - K)|\mathcal{O}(t))$ and $\mathcal{L}|\mathcal{O}(t))$ is equivalent with linear dependence between $e^{-it\mathcal{L}}(\mathcal{K} - K)e^{it\mathcal{L}}\mathcal{O}$ and $|\mathcal{O}_1)$. To simplify, we will use the notation $\mathbb{L}^n$ to mean $[\mathcal{L}, \cdot]$ applied to $\mathcal{K}$ n times. By Taylor expanding the vector $e^{-it\mathcal{L}}(\mathcal{K} - K)e^{it\mathcal{L}}|\mathcal{O})$ at $t = 0$, we have that

$$e^{-it\mathcal{L}}(\mathcal{K} - K)e^{it\mathcal{L}}|\mathcal{O}) = \Big(- K + \sum_{n=1}^{\infty} \frac{1}{n!}(-i)^n \mathbb{L}^n t^n \Big)|\mathcal{O}). \tag{20}$$

It is clear that $\mathbb{L} = \mathcal{L}_- - \mathcal{L}_+$ while $\mathbb{L}^2 = 2[\mathcal{L}_+, \mathcal{L}_-]$ is diagonal in the Krylov basis with eigenvalues $(\mathbb{L}^2)_{n,n} = -2(b_{n+1}^2 - b_n^2)$. Applying $[\mathcal{L}, \cdot]$ once more, one finds that $\mathbb{L}^3$ consists only of a subdiagonal and superdiagonal with values given by $(\mathbb{L}^3)_{n+1,n} = -(\mathbb{L}^3)_{n,n+1} = -2b_{n+1}f(n)$, where f is the discrete function defined by $f(n) = (b_{n+2}^2 - b_{n+1}^2) - (b_{n+1}^2 - b_n^2)$. By the k-diagonal of a matrix, we mean the diagonal of the matrix going top-left to bottom-right direction where k is an offset from the main diagonal. We use the convention that $k = 0$ is the main diagonal while $k = 1$ and $k = -1$ are the superdiagonal and subdiagonal respectively, and so on. From the form of $\mathbb{L}^3$, it should be clear that k-diagonals of $\mathbb{L}^{n+3}$ for which $|k| > 1 + n$ must only consist of zero-valued entries. Consequently, we must have that $(\mathbb{L}^{n+4})_{n+m+2,m} = [\mathcal{L}_+, \mathbb{L}^{n+3}]_{n+m+2,m}$ which more explicitly can be written as the recursion relation $(\mathbb{L}^{n+4})_{n+m+2,m} = b_{n+m+3}(\mathbb{L}^{n+3})_{n+m+1,m} - b_{m+1}(\mathbb{L}^{n+3})_{n+m+2,m+1}$. To simplify some notation, we will write $L(n,m) \equiv (\mathbb{L}^{n+4})_{n+m+2,m}$ and the recursion relation can then be written as $L(n,m) = b_{n+m+2}L(n-1,m) - b_{m+1}L(n-1,m+1)$ for $n > 0$.

We now observe that the following proposition must be true:

Proposition 1. *The condition: $L(n,0) = 0 \ \forall \ 0 \le n \le D-3$, is a necessary condition for the vector $e^{-it\mathcal{L}}(\mathcal{K} - K)e^{it\mathcal{L}}|\mathcal{O})$ to be linearly dependent of $|\mathcal{O}_1)$, and therefore, a necessary condition for the dispersion bound to be satisfied.*

By applying $[\mathcal{L}, \cdot]$ to $\mathbb{L}^3$, one finds that $L(0,m) = 2b_{m+1}b_{m+2}g(m)$, where we have defined $g(m) = f(m) - f(m+1)$. We will show that the condition $L(n,0) = 0 \ \forall \ 0 \le n \le D-3$ is equivalent to the complexity algebra being closed. Together with Proposition 1, this would then prove that the algebra being closed is a necessary condition for saturation of the dispersion bound. In order to prove this however, we will first prove another proposition.

Lemma 2. *Consider the discrete function $L(n,m)$ where $0 \le n \le D-3$ and $0 \le m \le D-1$. The recursion relation $L(n,m) = b_{n+m+2}L(n-1,m) - b_{m+1}L(n-1,m+1)$ together with the initial condition $L(0,m) = 2b_{m+1}b_{m+2}g(m)$ implies:*

$$L(n,m) = 2\prod_{j=1+m}^{2+n+m} b_j \sum_{k=0}^{n} (-1)^k \binom{n}{k} g(m+k). \tag{21}$$

Proof. We prove this by using mathematical induction. For the base case we have that

$$\begin{aligned} L(1,m) &= b_{m+3}L(0,m) - b_{m+1}L(0,m+1) \\ &= b_{m+3}\big(2b_{m+1}b_{m+2}g(m)\big) - b_{m+1}\big(2b_{m+2}b_{m+3}g(m+1)\big) \\ &= 2b_{m+1}b_{m+2}b_{m+3}\big(g(m) - g(m+1)\big). \end{aligned} \tag{22}$$

For the inductive step we have

$$\begin{aligned} L(n,m) &= b_{n+m+2}L(n-1,m) - b_{m+1}L(n-1,m+1) \\ &= 2\prod_{j=m+1}^{n+m+2} b_j \left(\sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} g(m+k) - \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} g(m+1+k) \right) \\ &= 2\prod_{j=m+1}^{n+m+2} b_j \left(\sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} g(m+k) + \sum_{k=1}^{n} (-1)^k \binom{n-1}{k-1} g(m+k) \right) \\ &= 2\prod_{j=m+1}^{n+m+2} b_j \left(g(m) + (-1)^n g(m+n) + \sum_{k=1}^{n-1} (-1)^k \left[\binom{n-1}{k-1} + \binom{n-1}{k} \right] g(m+k) \right) \\ &= 2\prod_{j=m+1}^{n+m+2} b_j \sum_{k=0}^{n} (-1)^k \binom{n}{k} g(m+k), \end{aligned} \tag{23}$$

where, in obtaining the second last line, we have made use of the binomial identity $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. ■

Corollary 3. $L(n,0) = 2\prod_{j=1}^{2+n} b_j \sum_{k=0}^{n} (-1)^k \binom{n}{k} g(k)$.

Proposition 4. $L(n,0) = 0 \ \forall \ 0 \le n \le D-3 \Leftrightarrow g(n) = 0 \ \forall \ 0 \le n \le D-3$.

Proof. We can think of the set of functions $g(n)$ as spanning a subset of $\mathbb{R}^{D-3}$. It should then be clear from Corollary 3 that the set of functions $L(n,0)$ must then have the same span. This means that we can express each $g(n)$ as a linear combination of the functions $L(n,0)$ or vice versa. Equating each function $L(n,0)$ ($g(n)$) with zero then results in $g(n) = 0$ ($L(n,0) = 0$) for all $0 \le n \le D-3$. ■

We are now ready to prove the following proposition:

Proposition 5. *The saturation of the dispersion bound implies that the complexity algebra is closed.*

Proof. We have that $\mathcal{B} \equiv \mathbb{L}$ and $\tilde{\mathcal{K}} \equiv \mathbb{L}^2$ and the complexity algebra is closed per definition if and only if $\mathbb{L}^3 = [\mathcal{L}, \tilde{K}]$ can only be written as a linear combination of $\mathcal{L}$, $\mathcal{B}$ and $\tilde{\mathcal{K}}$. It should be clear that this is possible if and only if $f(n) = C \ \forall\ 0 \leq n \leq D-2$, where $C \in \mathbb{R}$. This is clearly equivalent to the condition $g(n) = 0 \ \forall\ 0 \leq n \leq D-3$, which together with Proposition 1 and 4 is implied by saturation of the dispersion bound. ∎

Proving sufficiency

As we pointed out in the proof of Proposition 5, the complexity algebra being closed is equivalent with $f(n) = C \ \forall\ 0 \leq n \leq D-2$, where $C \in \mathbb{R}$. We note that

$$f(n) = C \ \forall\ 0 \leq n \leq D-2 \ \Leftrightarrow\ 2(b_{n+1}^2 - b_n^2) = \alpha n + \gamma \quad \forall\ 0 \leq n \leq D-1 \tag{24}$$

$$\Leftrightarrow\ b_n = \sqrt{\frac{1}{4}\alpha n(n-1) + \frac{1}{2}\gamma n + \delta} \quad \forall\ 0 \leq n \leq D, \tag{25}$$

where α, γ and δ are real constants and $C = \frac{1}{2}\alpha$. We stress that (25) holds under the convention that $b_0 = 0$, and we note that this implies that $\delta = 0$ and so we must have

$$b_n = \sqrt{\frac{1}{4}\alpha n(n-1) + \frac{1}{2}\gamma n} \quad \forall\ 1 \leq n \leq D. \tag{26}$$

The right hand side of the equivalence sign in (24) is equivalent to $\tilde{K} = \alpha\mathcal{K} + \gamma$. Consequently, the closed complexity algebra is entirely determined by the commutation relations $[\mathcal{K}, \mathcal{L}] = \mathcal{B}$, $[\mathcal{K}, \mathcal{B}] = \mathcal{L}$ and $[\mathcal{L}, \mathcal{B}] = \alpha\mathcal{K} + \gamma$.

Lemma 6. *The complexity algebra being closed implies that* $\mathbb{L}^{2n} = (-1)^n \frac{1}{\alpha}(\sqrt{\alpha})^{2n}(\alpha\mathcal{K}+\gamma)$ *and* $\mathbb{L}^{2n+1} = (-1)^{n+1}\frac{1}{\sqrt{\alpha}}(\sqrt{\alpha})^{2n+1}\mathcal{B}$ *when* $\alpha \neq 0$ *and* $\mathbb{L} = -\mathcal{B}$, $\mathbb{L}^2 = -\gamma$ *and* $\mathbb{L}^n = 0$ *for* $n > 2$ *when* $\alpha = 0$.

Proof. The case for when $\alpha = 0$ is trivial while for $\alpha \neq 0$ we will use mathematical induction. For the base case we have $\mathbb{L}^2 = [\mathcal{L}, [\mathcal{L}, \mathcal{K}]] = -[\mathcal{L}, \mathcal{B}] = -(\alpha\mathcal{K} + \gamma)$ and $\mathbb{L}^3 = [\mathcal{L}, -(\alpha\mathcal{K} + \gamma)] = \alpha\mathcal{B}$. For the inductive step, we have $\mathbb{L}^{2n} = [\mathcal{L}, [\mathcal{L}, \mathbb{L}^{2(n-1)}]] = (-1)^{n-1}(\sqrt{\alpha})^{2(n-1)}[\mathcal{L}, [\mathcal{L}, \mathcal{K}]] = (-1)^n \frac{1}{\alpha}(\sqrt{\alpha})^{2n}(\alpha\mathcal{K}+\gamma)$ and $\mathbb{L}^{2n+1} = [\mathcal{L}, \mathbb{L}^{2n}] = (-1)^{n+1}\frac{1}{\sqrt{\alpha}}(\sqrt{\alpha})^{2n+1}\mathcal{B}$. ∎

Proposition 7. *The complexity algebra being closed implies that the dispersion bound is saturated.*

Proof. By Lemma 6 we have $\mathbb{L}^{2n} = (-1)^n \frac{1}{\alpha}(\sqrt{\alpha})^{2n}(\alpha\mathcal{K} + \gamma)$ and $\mathbb{L}^{2n+1} = (-1)^{n+1}\frac{1}{\sqrt{\alpha}}(\sqrt{\alpha})^{2n+1}\mathcal{B}$ when $\alpha \neq 0$. By substituting these into the Taylor expansion of $e^{-it\mathcal{L}}(\mathcal{K} - K)e^{it\mathcal{L}}|\mathcal{O})$, we have

$$\begin{aligned} e^{-it\mathcal{L}}(\mathcal{K} - K)e^{it\mathcal{L}}|\mathcal{O}) &= \Big(-K + \sum_{n=1}^{\infty} \frac{1}{n!}(-i)^n \mathbb{L}^n t^n \Big)|\mathcal{O}) \\ &= \Big(-K + \sum_{n=1}^{\infty} \frac{1}{(2n)!}(-i\mathbb{L}t)^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(-i\mathbb{L}t)^{2n+1} \Big)|\mathcal{O}) \\ &= \Big(-K + \frac{\gamma}{\alpha}\sum_{n=1}^{\infty} \frac{1}{(2n)!}(\sqrt{\alpha}t)^{2n} + \frac{1}{\sqrt{\alpha}} i\mathcal{B} \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(\sqrt{\alpha}t)^{2n+1} \Big)|\mathcal{O}) \\ &= \Big(-K + \frac{\gamma}{\alpha}(\cosh\sqrt{\alpha}t - 1) + \frac{1}{\sqrt{\alpha}} i\mathcal{B}\sinh\sqrt{\alpha}t \Big)|\mathcal{O}). \end{aligned} \tag{27}$$

Since $\mathcal{B}|\mathcal{O}) = b_1|\mathcal{O}_1)$, it follows from the definition of Krylov complexity that the first two terms in the expression above must cancel. We thus have that

$$e^{-it\mathcal{L}}(\mathcal{K} - K)e^{it\mathcal{L}}|\mathcal{O}) = \frac{b_1}{\sqrt{\alpha}}\sinh\sqrt{\alpha}t|\mathcal{O}_1). \tag{28}$$

When $\alpha = 0$, one has that $\mathbb{L} = -\mathcal{B}$, $\mathbb{L}^2 = -\gamma$ and $\mathbb{L}^n = 0$ for $n > 2$. Substituting these into the Taylor expansion, one finds that

$$\begin{aligned} e^{-it\mathcal{L}}(\mathcal{K} - K)e^{it\mathcal{L}}|\mathcal{O}) &= \Big(- K - i\mathbb{L}t - \frac{1}{2}\mathbb{L}^2 t^2\Big)|\mathcal{O}) \\ &= \Big(- K + \frac{\gamma}{2}t^2 + i\mathcal{B}t\Big)|\mathcal{O}) = ib_1|\mathcal{O}_1). \end{aligned} \tag{29}$$

We thus have that the algebra being closed is a sufficient requirement for saturating the dispersion bound. ■

The proofs of Proposition 5 and 7 leads to the conclusion that saturation of the dispersion bound is equivalent with the complexity algebra being closed.

Remark 8. *We would like to point out that equation* (27) *and* (29) *shows that the general solution for Krylov complexity, whenever the dispersion bound is saturated, is given by* $K(t) = -\frac{2\gamma}{\alpha}\sin^2\frac{\sqrt{-\alpha}t}{2}$ *when* $\alpha < 0$, $K(t) = \frac{\gamma}{2}t^2$ *when* $\alpha = 0$ *and* $K(t) = \frac{2\gamma}{\alpha}\sinh^2\frac{\sqrt{\alpha}t}{2}$ *when* $\alpha > 0$. *These three scenarios correspond to the three algebraic models discussed above:* $SL(2,\mathbb{R})$, *HW and* $SU(2)$ *respectively.*

Remark 9. *The requirement that* $b_n \geq 0$ *for all* n *implies that* $\gamma \geq 0$ *and* $\alpha \geq -\frac{2}{n-1}\gamma$ *for all* n. *In the infinite dimensional case we see that this implies that* $\alpha \geq 0$. *In the finite-dimensional case, the condition* $b_D = 0$ *implies that* $\alpha = -\frac{2}{D-1}\gamma$ *and so the solution of Krylov complexity only depends on* γ, *namely* $K(t) = (D-1)\sin^2\sqrt{\frac{\gamma}{2(D-1)}}t$. *By setting* $\omega = \sqrt{\frac{\gamma}{2(D-1)}}$ *we have that* $K(t) = (D-1)\sin^2\omega t$ *and the Lanczos coefficients grow according to* $b_n = \omega\sqrt{n(D-n)}$. *Therefore, by comparison with Eq.* (4), *we see that, in a finite* D*-dimensional Krylov space, the saturation of the bound for each time can be achieved only when the dynamics is governed, up to a multiplicative constant, by the* $SU(2)$ *algebra, in the representation* $j = \frac{D-1}{2}$.

Supplementary References

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