Synchronization induced by directed higher-order interactions
Supplementary Information

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SUPPLEMENTARY NOTE 1. SYNCHRONIZATION IN SYMMETRIC HYPERGRAPHS

The stability analysis presented in the main text for directed hypergraphs also applies to undirected hypergraphs (this latter case can also be seen as an extension of the method presented by Gambuzza et al. [1] developed for simplicial complexes), so that we here briefly discuss an example of synchronization in the presence of undirected higher-order interactions. Notice that, at variance with the derivation outlined in Methods, in the undirected case the adjacency tensor is symmetric, as the generalized Laplacian matrix of order d does. This latter is in fact given by

$$L_{ij}^{(d)} = \begin{cases} d!k^{(d)}(i) & i = j \\ -(d-1)!k^{(d)}(i,j) & i \neq j \end{cases} . \tag{S1}$$

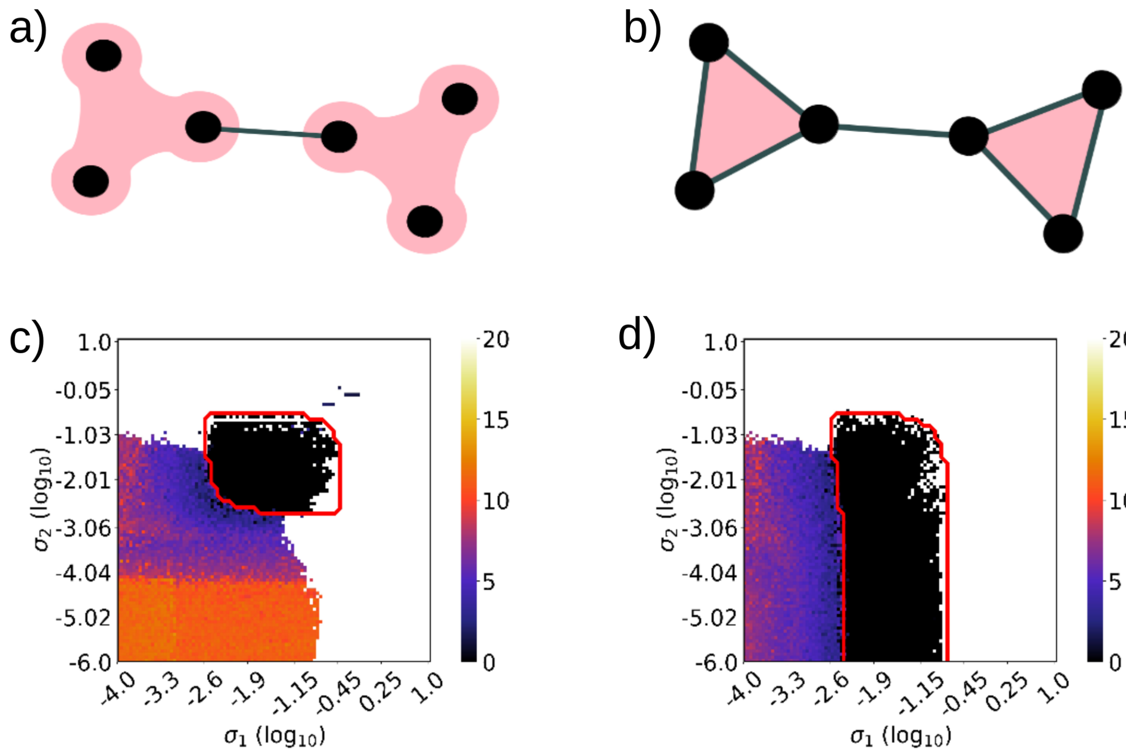
To illustrate our results, we consider again a system of N coupled Rössler oscillators, whose parameters have been set to $a = b = 0.2$, and $c = 9$, so that the dynamics of the isolated system is chaotic. The system is coupled via the x component, through the coupling functions $\mathbf{h}^{(1)}(\mathbf{x}_j) = [x_j^3, 0, 0]$ and $\mathbf{h}^{(2)}(\mathbf{x}_j, \mathbf{x}_k) = [x_j^2 x_k, 0, 0]$. The equations governing the system read

$$\begin{cases} \dot{x}_i = -y_i - z_i + \sigma_1 \sum_{j=1}^{N} A_{ij}^{(1)}(x_j^3 - x_i^3) + \sigma_2 \sum_{j,k=1}^{N} A_{ijk}^{(2)}(x_j^2 x_k - x_i^3) \\ \dot{y}_i = x_i + a y_i \\ \dot{z}_i = b + z_i(x_i - c), \end{cases} \tag{S2}$$

with $i \in \{1, \dots, N\}$.

In particular, we analyzed the undirected hypergraph with $N = 6$ nodes shown in panel a) of Supplementary Fig. 1. Note that no link exists between the nodes in the 2-hyperedges, meaning that the higher-order structure is not a simplicial complex. We simulate Eqs. (S2) on top of this structure, for different value of the coupling strengths σ_1 and σ_2. The state of the system is monitored by the average synchronization error defined as in Eq. (15) of the main text. Panel c) of Supplementary Fig. 1 displays the synchronization error $E(\sigma_1, \sigma_2)$ (colormap), along with the theoretical prediction of the boundary of the stability region provided by the MSF (solid red line). As one can see, the numerical simulations are in very good agreement with the theoretical predictions for the synchronization thresholds.

To fully appreciate the difference between (undirected) hypergraphs and simplicial complexes, let us consider a simplicial complex having the same 2-hyperedges as the structure in panel a) of Supplementary Fig. 1, but different links, as shown in panel b). Comparing panel c) with panel d), illustrating the synchronization error, one can conclude that the presence of pairwise interactions in the simplicial complex preserves the stability of the synchronized state even when the higher-order coupling σ_2 is small, while in the case of the hypergraph, under such conditions, synchronization is lost.



SUPPLEMENTARY FIGURE 1. Synchronization in hypergraphs and in simplicial complexes. In panel a) we depict an undirected hypergraph, in which two 2-hyperedges are connected through a pairwise link (black line), while in panel b) an undirected simplicial complex formed by two 2-simplices connected through a pairwise link (black line) is reported; each 2-simplex contains also the three pairwise interactions (black lines on the boundary of the triangles). This determines the key difference between hypegraphs and simplicial complexes. In panels c) and d), we report the averaged synchronization error $E(\sigma_1, \sigma_2)$ as defined in Eq. (15) of the main text by using the shown color code, with the solid red line depicting the theoretical prediction of the boundary of the stability region provided by the Master Stability Function.

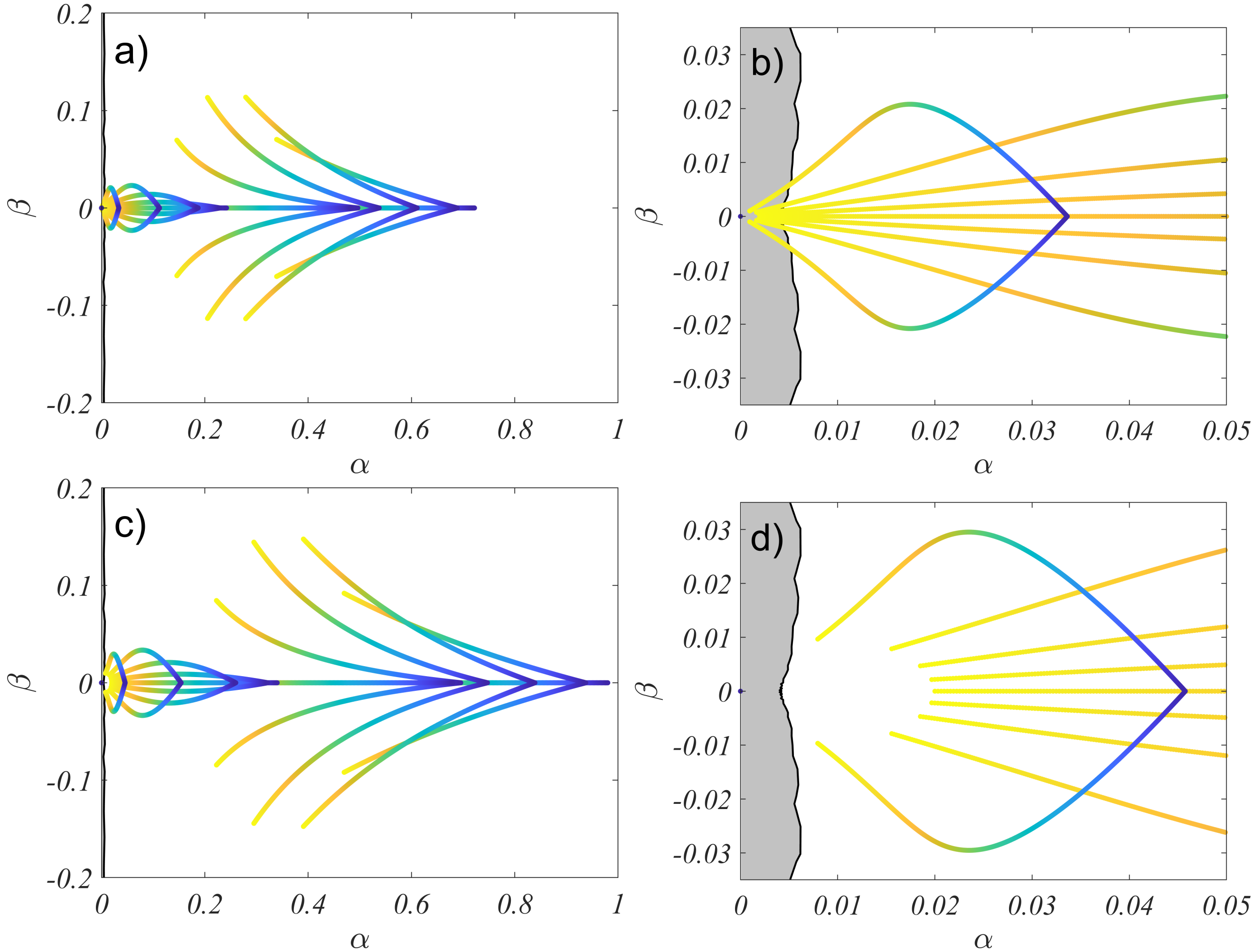
SUPPLEMENTARY NOTE 2. DIRECTED HYPERGRAPH OF RÖSSLER SYSTEMS WITH $y - y$ COUPLING

In the main text, we have considered a system of Rössler oscillators coupled through a 1-directed hypergraph, with the coupling functions being $\mathbf{h}^{(1)}(\mathbf{x}_j) = [x_j^3, 0, 0]$ and $\mathbf{h}^{(2)}(\mathbf{x}_j, \mathbf{x}_k) = [x_j^2 x_k, 0, 0]$. As a further example, we here account for a different choice of the coupling functions, namely $\mathbf{h}^{(1)}(\mathbf{x}_j) = [0, y_j^3, 0]$ and $\mathbf{h}^{(2)}(\mathbf{x}_j, \mathbf{x}_k) = [0, y_j^2 y_k, 0]$, which also satisfy the natural coupling hypothesis. The equations for the coupled system read

$$\begin{cases} \dot{x}_i = -y_i - z_i \\ \dot{y}_i = x_i + a y_i + \sigma_1 \sum\limits_{j=1}^{N} A_{ij}^{(1)} (y_j^3 - y_i^3) + \sigma_2 \sum\limits_{j,k=1}^{N} A_{ijk}^{(2)} (y_j^2 y_k - y_i^3) \\ \dot{z}_i = b + z_i (x_i - c). \end{cases} \tag{S3}$$

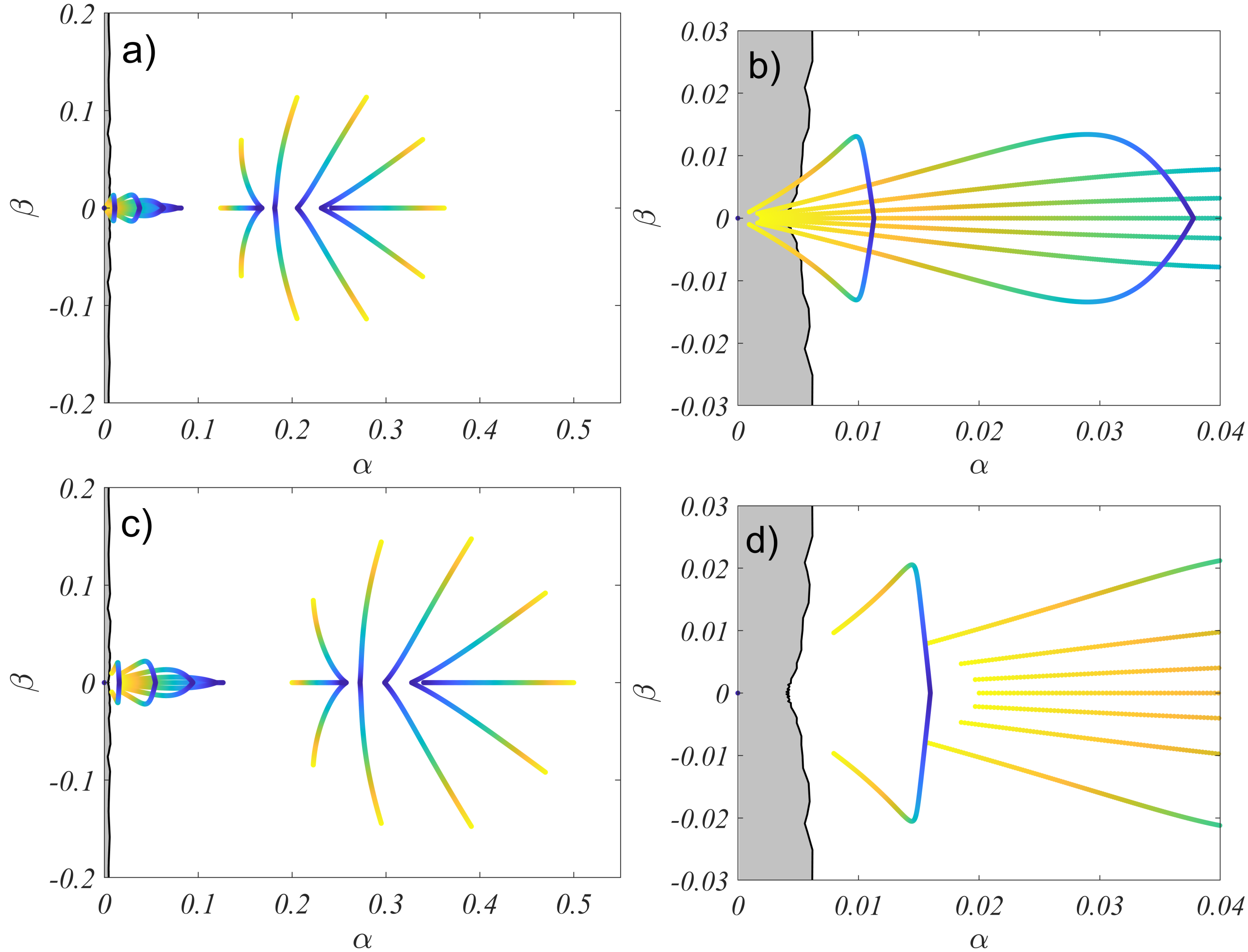
for $i = \{1, \dots, N\}$.

As we have done for the coupling on the x component, we study the effects of directed topology on synchronization by varying the directionality of the 2-hyperedges. Supplementary Fig. 2 shows the variation of the eigenvalues of $\mathcal{M}$ as a function of p for two different sets (σ_1, σ_2), namely $\sigma_1 = 0.001$ and $\sigma_2 = 0.12$, in panels a) and b), and $\sigma_1 = 0.01$ and $\sigma_2 = 0.16$, in panels c) and d). Observe that panels b) and d) represent a zoom of the area close to the origin in panels a) and c), respectively. In the background in each panel, the MSF for system (S3) is represented. In particular, the gray area represents the region where the MSF is positive, the white area portrays the region of stability, while the black line denotes the boundary value $\Lambda_{\max}(\alpha + \mathrm{i}\beta) = 0$. From the Figure, it can be noted that the shape of the MSF in this setting allows the system to go unstable only for low values of the parameter p, in contrast to the case shown in the main text. Panels a) and b) show the case where the directed hypergraph ($p = 0$) leads to the desynchronization



SUPPLEMENTARY FIGURE 2. Directionality induced (de)synchronization with y-y cubic coupling. Panels a)-d) show the locus of the eigenvalues of the effective Laplacian matrix $\mathcal{M}$ as a function of the symmetry parameter $p \in [0, 1]$, for a weighted hypergraph with $N = 20$ nodes (color coding is such that the directed case $p = 0$ is represented in yellow, and the symmetric one $p = 1$, in blue). In the background, the white area indicates the region identified by a negative Master Stability Function (MSF), the black line the boundary of this region, and the gray area the region where the MSF is positive. Panels b) and d) represent a zoom of the area close to the origin of panels a) and c), respectively. Panels a) and b) show how the directed topology drives the system unstable, while for the symmetric hypergraph the synchronization manifold is stable. Panels c) and d) display how, admitting the directed topology a stable synchronization state, giving the shape of the MSF, it is not possible to desynchronize the system of oscillators by making the hypergraph symmetric. The coupling strength of the two-body and the three-body interactions for panels a) and b) are $\sigma_1 = 0.001$ and $\sigma_2 = 0.12$, while for panel c) and d) they are fixed to $\sigma_1 = 0.01$ and $\sigma_2 = 0.16$.

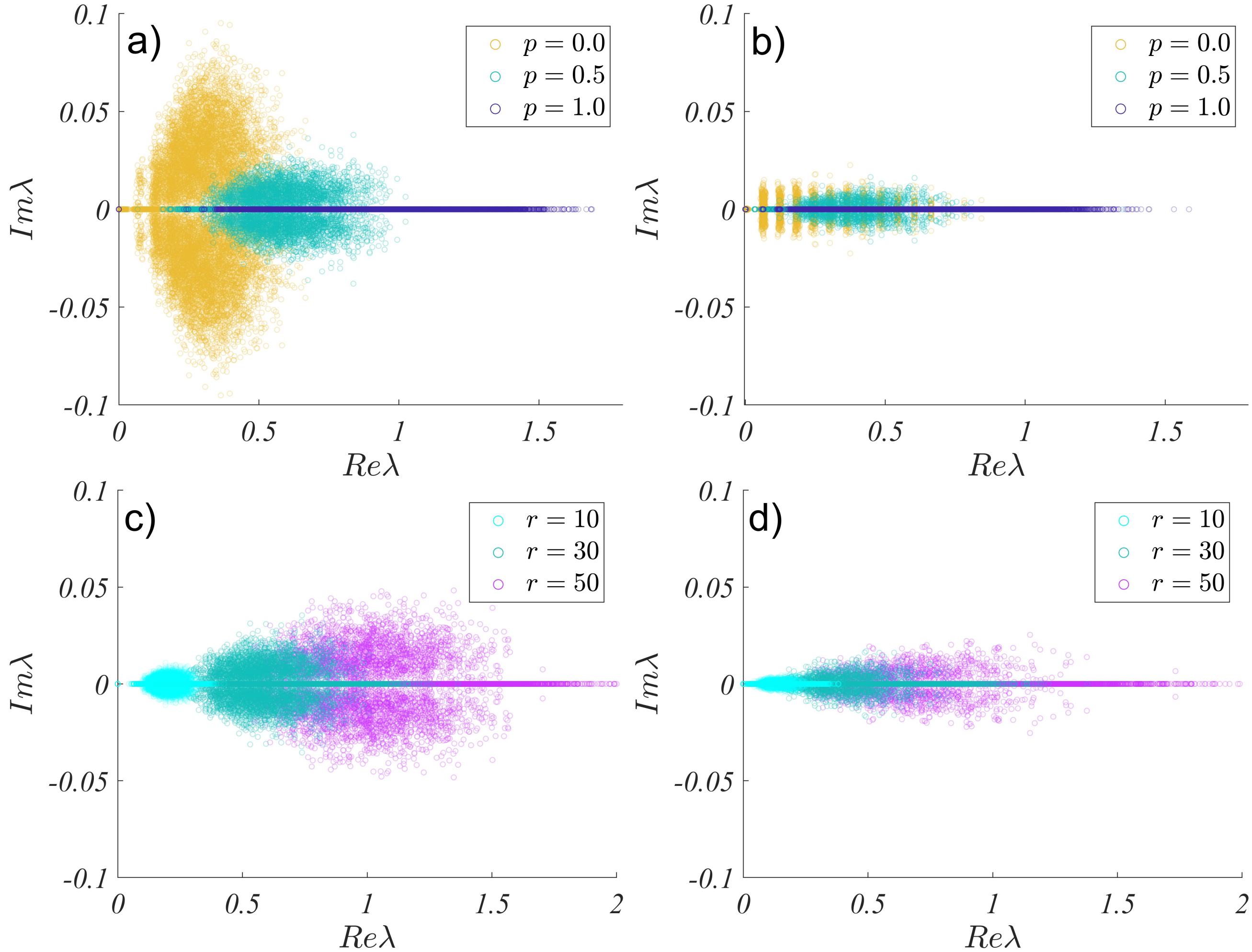
of the system, while the symmetric structure ($p = 1$) admits a stable solution. On the other hand, panels c) and d) display the case where, starting from a synchronous state that is stable for $p = 0$, by varying the value of p the eigenvalues remain in the area of the complex plane for which the MSF is negative. This means that, given the shape of the MSF, it is not possible to desynchronize the system by making the interactions among triplets of nodes more symmetric. A qualitatively similar behavior is obtained for a system where the method of symmetrization preserving the total coupling strength is applied. In Supplementary Fig. 3, we show that the synchronous state can be unstable when the higher-order topology is directed ($q = 0$) and stabilize as symmetry increases ($q \to 1/3$), while if the former is already stable, due to the shape of the MSF, the stability is preserved during the symmetrization.



SUPPLEMENTARY FIGURE 3. Directionality induced (de)synchronization with y-y cubic coupling, considering the alternative symmetrization method. Panels a)-d) display the locus of the eigenvalues of the effective Laplacian $\mathcal{M}$ as a function of the symmetry parameter $q \in [0, 1/3]$, for a weighted hypergraph with $N = 20$ nodes (color coding is such that the directed case $q = 0$ is represented in yellow, and the symmetric one $q = 1/3$, in blue). In the background, the white area indicates the region identified by a negative Master Stability Function (MSF), the black line the boundary of this region, and the gray area the region where the MSF is positive. Panels b) and d) represent a zoom of the area close to the origin of panels a) and c), respectively. Panels a) and b) show that directionality can drive the system unstable, while for the symmetric hypergraph the synchronous state is achieved. Panels c) and d) show that, provided that the directed topology admits a stable synchronization state and given the shape of the MSF, the system of oscillators does not lose synchronization as the hypergraph is made symmetric. The coupling strengths of the two-body and three-body interactions for panels a) and b) are $\sigma_1 = 0.001$ and $\sigma_2 = 0.12$, while for panel c) and d) they are fixed to $\sigma_1 = 0.01$ and $\sigma_2 = 0.16$.

SUPPLEMENTARY NOTE 3. SYNCHRONIZATION IN RANDOM HIGHER-ORDER STRUCTURES

Once fixed the MSF, it is the structure of the interactions that, determining the matrix $\mathcal{M}$, and so its eigenvalues, ultimately controls how directionality will impact synchronization. To investigate how the emerging dynamics is connected to the higher-order structure, here we analyze and compare two models for generating random hypergraphs. First, we consider a higher-order structure inspired by the Newman-Watts (NW) model [2]. In particular, we start from an undirected nonlocal ring of N nodes, where each unit is connected to its m nearest neighbors. Then, for each couple of nodes in the network we add a 1-directed 2-hyperedge pointing to a third randomly chosen node with probability ϕ. Second, we take into account a hypergraph version of the Erdős-Rényi (ER) model [3] ruled by two parameters. The first, as in the classical ER model for networks is the probability ρ_1 of connecting two nodes with



SUPPLEMENTARY FIGURE 4. Eigenvalue distribution of random higher-order structures. a) Spectrum distributions for the higher-order Newman-Watts model as a function of the symmetry parameter p. b) Distributions for the higher-order Erdős-Rényi model as a function of p. c) Distributions for the higher-order Newman-Watts model as a function of the ratio $r = \sigma_2/\sigma_1$, being σ_1 and σ_2 the coupling strength of the two-body and three-body interactions, respectively. d) Variation of the distributions for the higher-order Erdős-Rényi model as a function of the ratio r. In all cases, $\sigma_1 = 0.001$.

an undirected link, while the second is the probability ρ_2 of adding a 1-directed 2-hyperedge among three nodes.

As the hypergraphs are randomly generated, then the spectrum of the associated matrices $\mathcal{M}$ is also stochastic. Therefore, to understand how synchronization is affected by the hypergraph structure, we need to characterize how the eigenvalues are distributed in the complex plane as a function of the model parameters. In particular, we explore how the spectra of the random hypergraphs vary as a function of the symmetry parameter p and of the ratio $r = \sigma_2/\sigma_1$. In addition, as we also aim at comparing the spectra obtained using the two generative models, we set the model parameters so that the average number of links and the average number of 2-hyperedges connected to each node are the same for the two algorithms. For each parameter set and for each model, we evaluate the spectrum distribution over $I = 1000$ realizations of the hypergraphs.

Supplementary Fig. 4 displays the eigenvalues of $\mathcal{M}$ as a function of p (panels a) and b)) and r (panels c) and d)) for both the NW (a) and c)) and the ER (b) and d)) higher-order generalization. Typically for $p = 0$ there are eigenvalues with nonzero imaginary part such that they are spread into the I and IV quadrants of the complex plane, while these distributions shrink to the real axis for $p = 1$ (here we have set $r = 30$). We observe that the imaginary part in the NW-like model is generally larger compared to the one of the ER-like model. Similar results are obtained when varying the value of the ratio r. For this case, we note that the distributions of eigenvalues remain close to the real axis for small values of r, while they spread over the imaginary axis for larger values of r (here we set $p = 0.5$). Varying r, consistently with what observed above for a fixed value of this parameter, confirms that the eigenvalues in the NW-like model typically have a larger imaginary part compared to their counterparts in the ER-like model.

A comprehensive analysis of how the topological features of a higher-order structure impact on synchronization would require to find the conditions for which the eigenvalues of $\mathcal{M}$ are entirely contained in the stability region. A similar problem appears in the context of pairwise interactions, when directed interactions are considered. Some attempts to elucidate the relationship between eigenvalues of an asymmetric matrix and the emerging synchronous dynamics have been made by Hwang et al. [4], but the problem is still open. In the case of higher-order structures, this problem is even more complex as the matrix $\mathcal{M}$ includes contributions from a series of different Laplacian matrices.

SUPPLEMENTARY REFERENCES

[1] L. V. Gambuzza, F. Di Patti, L. Gallo, S. Lepri, M. Romance, R. Criado, M. Frasca, V. Latora, and S. Boccaletti, Stability of synchronization in simplicial complexes, Nature communications **12**, 1 (2021).

[2] M. E. Newman and D. J. Watts, Scaling and percolation in the small-world network model, Physical review E **60**, 7332 (1999).

[3] P. Erdős and A. Rényi, On the evolution of random graphs, Publ. Math. Inst. Hung. Acad. Sci **5**, 17 (1960).

[4] D.-U. Hwang, M. Chavez, A. Amann, and S. Boccaletti, Synchronization in complex networks with age ordering, Physical review letters **94**, 138701 (2005).