Supplementary Information: Non-inertial quantum clock frames lead to non-Hermitian dynamics

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SUPPLEMENTARY NOTE 1: DERIVATION OF EFFECTIVE HAMILTONIANS

Here, we provide details for the derivation of the Hamiltonians in Eqs. (11), (14), and (15).

First, we derive H_{eff}^A in Eq. (11). Defining $|\psi(t_A)\rangle \equiv \langle t_A|\Psi\rangle\rangle$ and using Eqs. (2), (3), and (9), we obtain

$$\begin{aligned}
0 &= \langle t_A|H|\Psi\rangle\rangle \\
&= \langle t_A| \left\{ [I + f(X_M)]\, H_A + H_M \right\} |\Psi\rangle\rangle \\
&= [I + f(X_M)] \left(\langle t_A|H_A \right) |\Psi\rangle\rangle + H_M \langle t_A|\Psi\rangle\rangle \\
&= [I + f(X_M)] \left(-i\hbar \frac{\partial}{\partial t_A} \langle t_A| \right) |\Psi\rangle\rangle + H_M |\psi(t_A)\rangle \\
&= -i\hbar\, [I + f(X_M)]\, \frac{\partial}{\partial t_A} |\psi(t_A)\rangle + H_M |\psi(t_A)\rangle.
\end{aligned} \tag{S1}$$

Assuming that $I + f(X_M)$ is invertible and reorganizing the terms, we obtain Eq. (10) with H_{eff}^A given by Eq. (11). In the case we are considering, we assume the particle is moving in a single direction since its acceleration can be parameterized by its position. Then, it should be possible to use systems of coordinates such that $I + f(X_M)$ is invertible everywhere. Yet, it should be noted that, whenever this expression is not invertible, an equation similar to Eq. (10) continues to hold. However, in this expression, the term $I + f(X_M)$ should remain multiplying the time derivative.

Now, a similar derivation leads to the Hamiltonian in Eq. (14). Defining $|\psi(t_A)\rangle \equiv \langle t_A|\Psi\rangle\rangle$ and using Eqs. (2), (3), and (13), we obtain

$$\begin{aligned}
0 &= \langle t_A| \left\{ [I + f(X_N, H_B)]H_A + H_B + H_M + H_N \right\} |\Psi\rangle\rangle \\
&= [I + f(X_N, H_B)] \left(\langle t_A|H_A \right) |\Psi\rangle\rangle + H_B \langle t_A|\Psi\rangle\rangle + H_M \langle t_A|\Psi\rangle\rangle + H_N \langle t_A|\Psi\rangle\rangle \\
&= [I + f(X_N, H_B)] \left(-i\hbar \frac{\partial}{\partial t_A} \langle t_A| \right) |\Psi\rangle\rangle + \left(H_B + H_M + H_N \right) |\psi(t_A)\rangle \\
&= -i\hbar\, [I + f(X_N, H_B)]\, \frac{\partial}{\partial t_A} |\psi(t_A)\rangle + \left(H_B + H_M + H_N \right) |\psi(t_A)\rangle.
\end{aligned} \tag{S2}$$

Assuming that $I + f(X_N, H_B)$ is invertible and reorganizing the terms, we obtain Eq. (10) with H_{eff}^A given by Eq. (14).

Finally, the same reasoning leads to Eq. (15). It is just necessary to observe that the Hamiltonian H can be also written as

$$H = H_A + [I + f(X_N, H_A)]H_B + H_M + H_N. \tag{S3}$$

SUPPLEMENTARY NOTE 2: EXAMPLE OF NON-UNITARY DYNAMICS WITH IDEAL CLOCK

Here, we show that the Hamiltonian in Eq. (5) with H_{int} given by Eq. (17) is generally non-unitary. For simplicity, we assume the clock is ideal. In this case, we first observe that

$$\int dt_A g(t_A) |t_A\rangle\langle t_A| = g(T_A). \tag{S4}$$

Then, we can write

$$H_{\rm int} = \frac{1}{2}\left[H_A g(T_A) + g(T_A) H_A\right]. \tag{S5}$$

Since the pair T_A and H_A is analogous to position and momentum for ideal clocks, we have $[H_A, g(T_A)] = -i\hbar g'(T_A)$, where g was assumed to be an analytic function and g', its first derivative. As a result,

$$H_{\rm int} = g(T_A)H_A - \frac{i\hbar}{2} g'(T_A). \tag{S6}$$

Then, recalling that $T_A|t_A\rangle = t_A|t_A\rangle$ for ideal clocks, Eq. (7) in the case of interest becomes

$$\begin{aligned} i\hbar\frac{\partial}{\partial t_A}|\psi(t_A)\rangle\rangle &= H_M|\psi(t_A)\rangle\rangle + \int dt'_A \; K(t_A, t'_A)|\psi(t'_A)\rangle\rangle \\ &= H_M|\psi(t_A)\rangle\rangle + i\hbar\left[g(t_A)\frac{\partial}{\partial t_A} - \frac{1}{2}g'(t_A)\right]|\psi(t_A)\rangle\rangle. \end{aligned} \tag{S7}$$

Reorganizing the terms and assuming that $1 - g(t_A)$ is invertible we obtain the Hamiltonian in Eq. (10) with $H_{\rm eff}^A$ given by Eq. (18).

If $H_M = P_M^2/2m$ and the initial state of system M is $|\psi(0)\rangle\rangle = (2\Delta^2/\pi)^{1/4}\int dp_M \; e^{-\Delta^2 p_M^2}|p_M\rangle$, we have

$$|\psi(t_A)\rangle\rangle = \left(\frac{2\Delta^2}{\pi}\right)^{1/4} e^{-\int_0^{t_A} dt'_A \; [1-g(t'_A)]^{-1} g'(t_A)/2} \int dp_M \; e^{-i\int_0^{t_A} dt'_A \; [1-g(t'_A)]^{-1} p_M^2/2m\hbar} e^{-\Delta^2 p_M^2}|p_M\rangle \tag{S8}$$

and, moreover, the result in Eq. (19).