

Supplementary Information for: Effect of Aberration on 3D optical topologies

Supplementary Note 1. EXPLICIT EXPRESSIONS OF THE KNOT FIELDS.

In the following, we give expressions for the linked and knotted fields in terms of Milnor polynomials modulated by a Gaussian envelope. We chose a , b and s in such a way as to obtain the cleanest singular skeleton in the not aberrated case. The Hopf-link was generated with a mask imposing, on the SLM's image plane, the field amplitude

$$\begin{aligned} \psi_{\text{Hopf link}}(\rho, \phi) = & e^{-\frac{\rho^2}{2s^2}} (1 - 2(1 + a^2 - b^2)\rho^2 + \rho^4 \\ & - 2(a^2 + b^2)\rho^2 \cos(2\phi) - 4iab\rho^2 \sin(2\phi)) \end{aligned} \quad (\text{S1})$$

with $a = 1, b = 1$ and $s = 1.7$. Similarly, the trefoil field was generated from the propagation of the amplitude:

$$\begin{aligned} \psi_{\text{Trefoil}}(\rho, \phi) = & e^{-\frac{\rho^2}{2s^2}} (-4\rho^3(a^2 - b^2) - 2\rho^3(a - b)^2 e^{-i3\phi} \\ & - 2\rho^3(a + b)^2 e^{i3\phi} + \rho^6 - \rho^4 - \rho^2 + 1) \end{aligned} \quad (\text{S2})$$

with $a = 1, b = 1$ and $s = 1.2$, while the cinquefoil was obtained from

$$\begin{aligned} \psi_{\text{cinquefoil}}(\rho, \phi) = & e^{-\frac{\rho^2}{2s^2}} (1 + \rho^2 - 2\rho^4 - 2\rho^6 + \rho^8 + \rho^{10} \\ & - 16\rho^5(a^2 - b^2) - 8\rho^5(a - b)^2 e^{-i5\phi} \\ & - 8\rho^5(a + b)^2 e^{i5\phi}) \end{aligned} \quad (\text{S3})$$

with $a = 0.5, b = 0.4$ and $s = 0.65$.

Supplementary Note 2. DATA ANALYSIS.

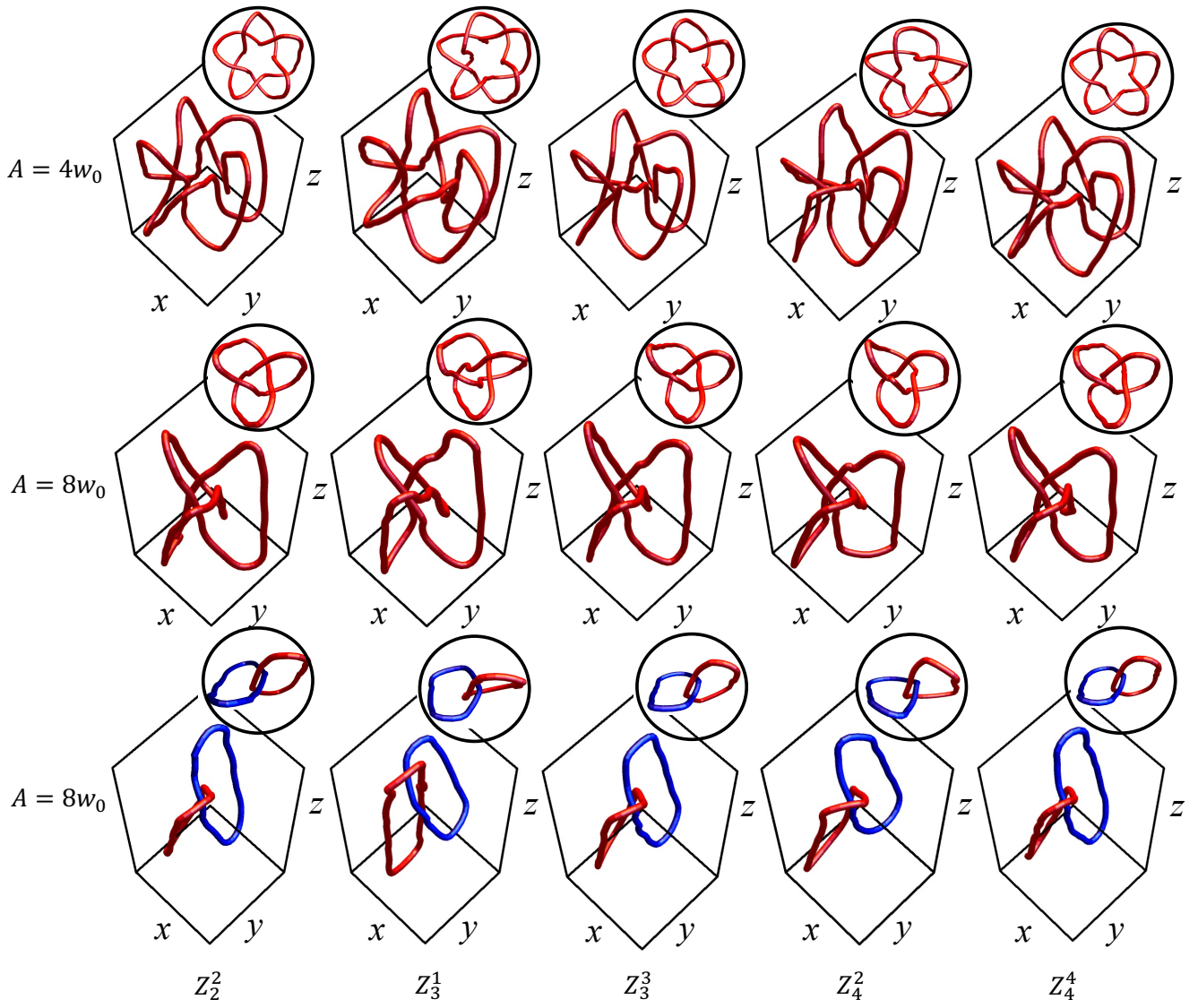
The experimental reconstruction of the singular skeleton was obtained by measuring, along the propagation direction z , the interference patterns $I_\alpha(z) = \left| \tilde{\psi}(\rho, \phi, z) + e^{i\alpha} \psi_{\text{ref}}(\rho, z) \right|^2$. From these measurements, the knotted field phase $\xi(\rho, \phi, z)$ (assuming a uniform phase for the reference beam) was retrieved as $\xi = \arg[I_0 - I_\pi + i(I_{\pi/2} - I_{3\pi/2})]$. The coordinates (x^*, y^*) of the phase singularities in each plane were extracted finding the intersections of the lines given by the equations $I_0(x, y) - I_\pi(x, y) = 0$ and $I_{\pi/2}(x, y) - I_{3\pi/2}(x, y) = 0$. The solution of these equations at different values of z yields a scatterplot of singular points which are then sorted and joined by a path minimization algorithm to reconstruct the singular skeleton.

Supplementary Note 3. SUPPLEMENTARY DATA.

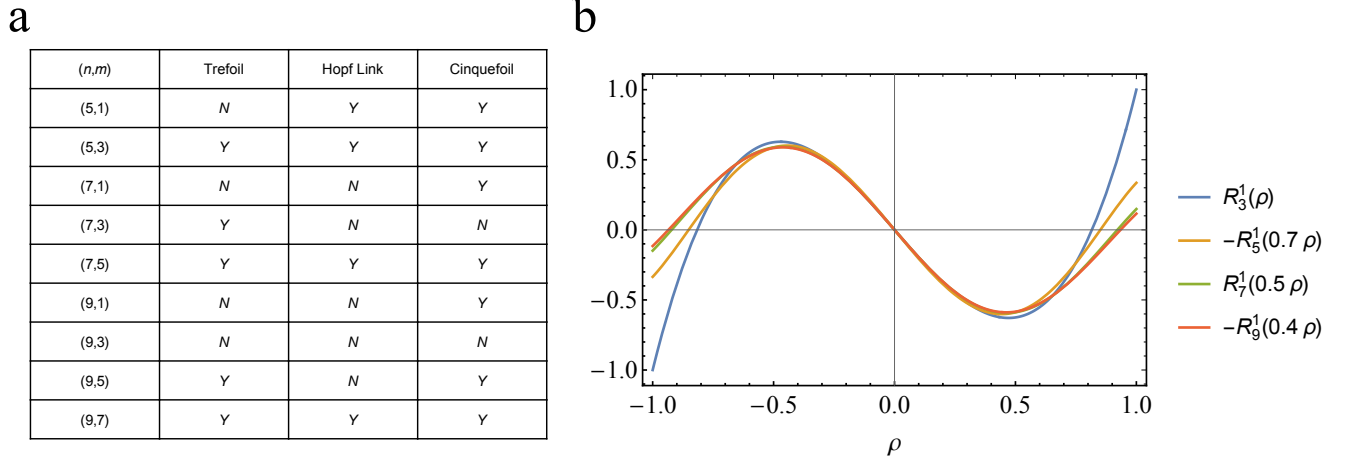
In Fig. 1, we show how for the apertures $A = 4w_0$ (for cinquefoil) and $A = 8w_0$ (for trefoil and Hopf link) the topology of the singular skeletons is not affected, up to $\gamma = 1$, by aberrations up to order 4.

In Fig. 2-a, we report results of simulations considering higher order aberrations. For $n > 4$ we can observe knot breakup when considering the same apertures and strength as in Fig. 1. When $m < 4$, these results can be interpreted as equivalent to shrinking the aperture of lower order aberrations –as illustrated in panel b for the case $m = 1$ –.

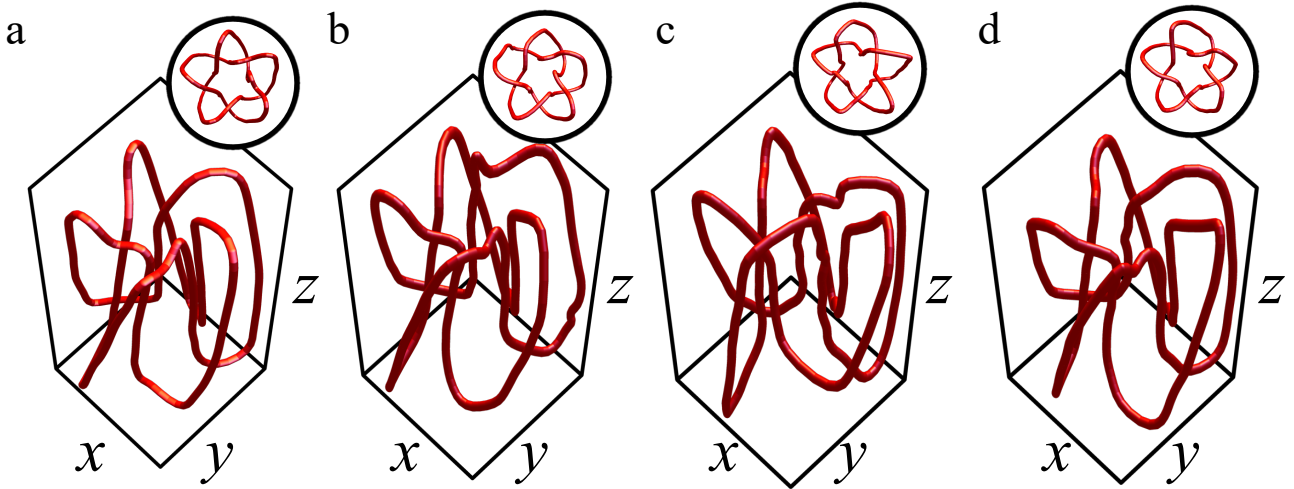
Figure 3 shows that a displacement of $\Delta x = w_0$ of the window defining the wavefront distortion does not change the topology of cinquefoil knots.



Supplementary Figure 1. Aberrated Knots and links. Additional examples of topological singular skeleton affected by aberrations with $\gamma = 1$ and aperture A . The rows represent, from top to bottom, cinquefoils, trefoils and Hopf links, with top views as insets. From left to right, the different singular skeletons were affected by, respectively, astigmatism, coma, trefoil, secondary astigmatism, and quadrafoil aberrations.



Supplementary Figure 2. Effect of aberrations of order $n > 4$ a. Summary of the effect of some higher order aberrations on the investigated optical topologies. the results were obtained with numerical simulations using the same parameters considered in the experiment for Zernike polynomials up to order 4. *Y* means that the topology survives the aberration (under strength $\gamma = 1$ and same apertures considered in the main text) while *N* means that the singular skeleton does not exhibit knotted or linked structures. In b we illustrate how the deleterious effect of higher order aberrations can be explained noticing how, for instance, the corresponding radial polynomials $R_n^1(\rho)$ have a very similar structure and a good overlap if re-scaling appropriately the coordinates. The considered aberrations behave indeed as a coma with a reduced aperture parameter.



Supplementary Figure 3. Cinquefoil knots under displaced aberrations. Experimentally reconstructed cinquefoil knots affected by different aberrations displaced by $\Delta x = w_0$ with defining aperture $A = 4w_0$, and strength $\gamma = 1$. The different panels correspond respectively to: a, astigmatism Z_2^2 , b, oblique trefoil Z_3^3 , c, quadrafoil Z_4^4 , and d, secondary astigmatism Z_4^2 .