nature portfolio

Peer Review File

Topological phase transitions of generalized brillouin zone

Reviewers' comments:

Reviewer #1 (Remarks to the Author):

This paper considers a non-Hermitian system with a generalized boundary condition that can continuously be tuned from a periodic to an open boundary condition. The authors first concentrate on arguably the simplest non-Hermitian model that exhibits the non-Hermitian skin effect, namely the Hatano-Nelson model. The authors then consider a dissipative XY model with asymmetric coupling. In both cases, they find a phase transition that is characterized by an exceptional point, similar to the one found in Ref. [10].

The results are interesting and the mechanism of transition they find is novel, to my knowledge. I, however, feel the physical consequence of the finite winding number W_non-Bloch they introduced from \tilde H_B is unclear (or at least was not clearly presented). I, therefore, recommend this manuscript for publication only after the following comments are answered.

I am quite confused about the physical meaning of the eigenvalues of \tilde H_B defined in Eq. (4) and the winding number associated with it. The relation
\tilde H_B (c_1, c_2)^T = (c_1, c_2)^T …(*),
is different from the conventional (non-Hermitian) Schrodinger equation where the left-hand side would be multiplied by the eigenvalue E in the latter. The matrix \tilde H_B is therefore not the Hamiltonian in the conventional sense. Unlike in the conventional case, I would expect the only eigenstate that has any physical meaning would be the one that has an eigenvalue 1 since otherwise it would not satisfy Eq. (*). Since the winding number W_non-Bloch that the authors introduced is NOT constructed from the eigenstates of a physically relevant operator, I would have imagined that W_non-Bloch defined from \tilde H_B would not have any physical meaning. Can the authors comment on this point? In particular, what is the physical consequence of having the non-zero winding number W_non-Bloch? For example, the winding number of the non-Hermitian Hamiltonian implies the emergence of the non-Hermitian skin effect or the edge state via the bulk-boundary correspondence. How about this case? (I do see a comment on p. 3 "The physical manifestation of Wnon−Bloch manifests as the topological bound state localized on the boundary of the chain" but please expand this part in the main text.)

The authors show that the phase rigidity vanishes at the transition point characterized by an exceptional point for GBC-(I), but how about other cases? Is this feature a unique feature for GBC-(I), or is this a generic property? Why or why not? I ask because if this phenomenon has topological origins, I will imagine this sort of singularity to be robust against the details of the GBCs.

Overall, I have the impression that the topological number the authors introduced might not be the right quantity to characterize the exceptional transition they have found for GBC-(I). I might be missing something, and it would be helpful if the authors could clearly give concrete evidence that my suspicion is not the case.

Minor comment:
What is {\mathcal L}_1 and {\mathcal L}_2? I believe they are not defined in the main text.

Reviewer #2 (Remarks to the Author):

In the manuscript "Topological Phase Transitions of Generalized Brillouin Zone", the authors proposed the general boundary conditions in non-Hermitian systems by adjusting the onsite and hopping strengths of the sample. With the help of the GBC, they demonstrated that a topological state is obtained. The topological states are accompanied by the appearance of some bound states. By adopting the winding number approach, they further point out that the topological phases transition can be captured by the variation of the Generalized Brillouin Zone. Finally, they extended their results to the Kuramoto model. In my opinion, their results are new and valid. The relationships between the proposed GBZ and the existence of the bound states looks rather interesting. However, I have several questions about the topological natures and the definitions of GBZ.

(1)The GBZ proposed in this manuscript seems totally different from the widely used ones. Generally, the GBZ is usually adopted to rebuild the bulk-boundary correspondences for samples under the open and periodic boundary conditions. The GBZ adopted in this manuscript is aimed to rewrite the eigenvalues in the "Hermitian" formula, i.e. $E\propto cos(z)$ instead of $E\propto [t_L exp(ik)+t_R exp(ik)]$ for the Hatano-Nelson model. Thus, the definition of the "GBZ" in this manuscript should be clarified more clearly in the main text.

(2)A topological systems should be robust against some-kinds of global disorders since topology is always regarded as a global concept. Taking the Hatano-Nelson model with Anderson disorders as an example (considering the on-site disorder:$\epsilon_i c_i^\dagger c_i$ with $\epsilon_i\in[-W/2,W/2]$), the NHSE and the related winding number still works. The robustness of the NHSE can be clearly identified as well. Thus, the NHSE is considered as the topological feature of non-Hermitian systems.

However, the definition of the GBC seems to rely on the translational symmetry. With the help of the translational symmetry, one is able to distinct the bulk and the boundary of the sample clearly. However, since the translational symmetry is destroyed everywhere after considering the global disorder [i.e. $\epsilon_i c_i^\dagger c_i$], the boundary could not be clearly defined. Based on the definition in the main text, every sites gives their GBC. So, does the topological natures still hold? The author should prove that the existence of the bound states are not originate from the local perturbation of the GBC.

(3)Some quantities are not clearly defined in the main text, i.e. the quantity "D=0.15" in the Fig. 4.

(4)I am a little bit confused about the meaning to pointing out the eigenstates ($\chi_{L/R}$) and eigenvalues (E) of the Eq.(4). If some quantities are pointed out, they should be adopted to clarify or demonstrate something.

Reviewer #3 (Remarks to the Author):

The present work investigates the intrinsic topology of the generalized Brillouin zone. It is shown that the topological boundary modes of the Hatano-Nelson model with the generalized boundary conditions can detect the topological phase transition in, where their wavenumber touche the generalized Brillouin zone and the exceptional points appear. The authors demonstrate that the nonreciprocal Kuramoto model exhibits the topological phase transition by showing the correspondence of the Hatano-Nelson model and the nonreciprocal Kuramoto model. They also propose that the phase transition accompanied by the appearance of the exceptional points results in the exceptional phase transition.

I think that the results obtained in this work are very interesting, because the topology of the single-particle description connects the phenomenon of many-body systems. Meanwhile, it seems that the descriptions of the terminologies and figures are lack. Hence, it would be good to elaborate on unclear points that I denote in the following, by adding the contents of the Method and Supplementary Material to the main text. If the authors appropriately address my points, I would like to recommend publishing the manuscript for Communications Physics.

(i) It is unclear how to read the figures in Fig. 2 because there are no explanations of Fig. 2 in the main text. For instance, what is the meaning of the vector plots in Figs. 2(a)-(c)? What is the difference between Fig. 2(a) and Fig. 2(b)? Where is the exceptional points in Fig. 2(b)? What is the definition of the phase rigidity?

(ii) I do not know that it is appropriate whether the discrete modes in Fig. 2(c) are called "topological boundary mode" or not. Conventionally, topological boundary modes appear in finite-size system because of the bulk-boundary correspondence. In contrast, the discrete modes appear in the system in the thermodynamic limit, as I refer to the Method. Thus, is there a conventional bulk-boundary correspondence in the topological phase predicted here?

(iii) I would like to ask the authors to clarify the role of the phase difference between the nearest-neighbor oscillators in the Kuramoto model because I do not know the detail of the model. Why are the boundary conditions determined by the phase difference?

(iv) It is unclear how to describe the limit-cycle phase in the Kuramoto model. It might be good to clarify the connection between this phase and the Jacobian. Furthermore, what is the definition of \partial_{i} in the Jacobian?

(v) I would like to the authors to add more explanation of the phase slip.

(vi) I think that the occurrence of the limit-cycle instability after the topological phase transition is interesting. To attract readers' attention, it would be good to visually clarify the formation of the positive real eigenvalue from the topological boundary state.

(vii) What is the meaning of the statement, "The physical consequence is the dynamical restoration of the spontaneously broken U(1) rotation symmetry"?

(viii) I do not know whether it is appropriate that Eq. (8) characterize the topological phase transition, or not. Does this mean that Eq. (8) is a topological invariant? If so, what is the difference between this quantity and the winding number proposed here?

Referee response for COMMSPHYS-23-0686-T

Dear reviewers,

We would like to thank the reviewers for taking the time to critically read and comment on our manuscript, and we are happy that the reviewers find our research interesting enough to proceed for next review. We have included our response to the comments and questions raised by the reviewers. Based on their feedback, we have modified the manuscript where appropriate.

Summary of changes in manuscript

- To answer the reviewer #1's comment, we have explained the physical significance of the non-Bloch topological invariant in the second paragraph (page 3, right column). We have also added another section (Sec. IE) in the supplementary material which contains an additional figure.
- To answer the reviewer #1's comment, we have elaborated on the definitions of GBZs in the second paragraph of page 3, left column.
- To answer the reviewer #2's comment, we have elaborated on the GBZs eigenvalue spectra in the first and second paragraph of page 3, left column.
- To answer the reviewer #2's comment, we have clarified the importance of chiral symmetry of the effective boundary matrix in the first paragraph of page 3, right column.
- To answer the reviewer #2's comment, we have defined the parameter in the caption of Fig. 4 (page 5) and in the main text in the first paragraph of page 6, left column.
- To answer the reviewer #3's comment, we have shown the locations of zeros and poles of the meromorphic function $\mathrm{h}_{\mathrm{B}}^{+}$, which determine the exceptional and topological phase transitions of GBZs in Fig. 2 (with little modification of the caption) and explained it in the text in the last paragraph (page 3, right column) and first paragraph (page 4, left column).
- To answer the reviewer #3's comment, we have added more explanations about the limit cycle phases, phase slips, Jacobian, and its relations to the Hatano-Nelson model in the second and third paragraphs (page 5, left column).
- To answer the reviewer #3's comment, we have elaborated on the dynamical restoration of the *U(1)*-symmetry in the second paragraph (page 5, right column).
- To answer the reviewer #3's comment, we have included the variation of the eigenvalue spectra in the time interval of the phase slip in Fig. 3 and accordingly modified its caption.

Response to Reviewer #1

Introductory comment

"This paper considers a non-Hermitian system with a generalized boundary condition that can continuously be tuned from a periodic to an open boundary condition. The authors first concentrate on arguably the simplest non-Hermitian model that exhibits the non-Hermitian skin effect, namely the Hatano-Nelson model. The authors then consider a dissipative XY model with asymmetric coupling. In both cases, they find a phase transition that is characterized by an exceptional point, similar to the one found in Ref. [10].

The results are interesting and the mechanism of transition they find is novel, to my knowledge. I, however, feel the physical consequence of the finite winding number $W_{non\text{-}Bloch}$ they introduced from H_B is unclear (or at least was not clearly presented). I, therefore, recommend this manuscript for publication only after the following comments are answered."

Response to introductory comment

We appreciate the referee for finding our work interesting. Also, we thank for raising important questions. Below, we address the referee's questions and make additions to the main text.

Comment 1

"I am quite confused about the physical meaning of the eigenvalues of H_B defined in Eq. (4) and the winding number associated with it. The relation $\mathrm{H_B}(c_1, c_2)^T = (c_1, c_2)^T$ ...(*), is different from the conventional (non-Hermitian) Schrodinger equation where the left-hand side would be multiplied by the eigenvalue E in the latter. The matrix $\mathrm{H_B}$ is therefore not the Hamiltonian in the conventional sense."

Response to comment 1

As the referee has mentioned, H_B is different from the conventional Hamiltonian. However, we point out that H_B has a physical meaning of scattering matrix, where the boundary plays a role of scatterer. Here, we briefly explain the key idea behind it. (The detailed analysis is given in Section E in the supplementary material)

We consider the generic wave function of incoming and outgoing wave in the presence of the potential barrier $(-L < x < L)$.

$$\psi(x) = \begin{cases} Ae^{iKx} + Be^{-iKx} & \text{when} \quad x < -L, \\ Ce^{iKx} + De^{-iKx} & \text{when} \quad x > L\,, \end{cases}$$

where $K \in \mathbb{C}$ can be a arbitrary complex number. The bound states are signatured by the poles of the reflection amplitude in the scattering matrix $(r = B/A, r' = C/D)$. Similarly in our lattice model, the off-diagonal components of the boundary matrix H_B plays a role of reflection amplitude: $\mathrm{h}_{\mathrm{B}}^{+}(z_1, z_2) \equiv -\frac{c_1}{c_2} = -\frac{A(z_2)}{A(z_1)} = -\frac{B(z_2)}{B(z_1)}$ for the wave function $\psi_n \sim c_1 z_1^n + c_2 z_2^n$ where H_{B} is given as,

$$H_{\mathrm{B}} = \begin{pmatrix} A(z_1) & A(z_2) \\ B(z_1) & B(z_2) \end{pmatrix}.$$

As in the case of the continuum model, the pole of the complex function $\mathrm{h}_{\mathrm{B}}^{+}(z_1, z_2)$ captures the bound states.

Comment 2

"Unlike in the conventional case, I would expect the only eigenstate that has any physical meaning would be the one that has an eigenvalue 1 since otherwise it would not satisfy Eq. (*). Since the winding number W_non-Bloch that the authors introduced is NOT constructed from the eigenstates of a physically relevant operator, I would have imagined that W_non-Bloch defined from H_{B} *would not have any physical meaning. Can the authors comment on this point? In particular, what is the physical consequence of having the non-zero winding number W_non-Bloch? For example, the winding number of the non-Hermitian Hamiltonian implies the emergence of the non-Hermitian skin effect or the edge state via the bulk-boundary correspondence. How about this case? (I do see a comment on p. 3 "The physical manifestation of Wnon−Bloch manifests as the topological bound state localized on the boundary of the chain" but please expand this part in the main text.)"*

<u>Response to comment 2</u>

Here, we explain the physical meaning of the winding number. The bound state is captured by the poles of $\mathrm{h}_{\mathrm{B}}^{+}(z_1, z_2)$ since it has physical meaning of the reflection amplitude in the complex momentum plane. If there is non-zero winding number in the phase of $\mathrm{h}_{\mathrm{B}}^{+}(z_1, z_2)$, there must exist the poles inside the integration contour. As a result, non-zero winding topological number in the reflection amplitude manifests as the topological boundary mode at the edge. In particular, the number of the winding number corresponds to the number of the boundary mode at the edge state.

We agree that the aforementioned comments are important for the potential audiences. The detailed explanation and information are now all included in the main text sand supplementary materials. In the main text, it reads as,

"The meromorphic function $\mathrm{h}_{\mathrm{B}}^{+}(z_1, z_2) \equiv -\frac{c_1}{c_2} = -\frac{A(z_2)}{A(z_1)} = -\frac{B(z_2)}{B(z_1)}$ are analogous to the reflection amplitudes for the scattering problem in quantum mechanics. Therefore, in order to have a bound state the pole and zero structure of the meromorphic function $\mathrm{h}_{\mathrm{B}}^{+}(z_1, z_2)$ has to change. Hence, the winding number $\mathrm{W}_{\text{non-Bloch}}$ defined for the meromorphic function $\mathrm{h}_{\mathrm{B}}^{+}(z_1, z_2)$ will detect the emergence of topological boundary states in the system (See Sec. IE of the supplementary material for more details).”

Comment 3

“The authors show that the phase rigidity vanishes at the transition point characterized by an exceptional point for GBC-(I), but how about other cases? Is this feature a unique feature for GBC-(I), or is this a generic property? Why or why not? I ask because if this phenomenon has topological origins, I will imagine this sort of singularity to be robust against the details of the GBCs. Overall, I have the impression that the topological number the authors introduced might not be the right quantity to characterize the exceptional transition they have found for GBC-(I). I might be missing something, and it would be helpful if the authors could clearly give concrete evidence that my suspicion is not the case.”

Response to comment 3

We admit that our manuscript is written in a confusing way. We make it clear that the vanishing phase rigidity is general feature that is observed for any kind of the boundary condition. We have explicitly shown this in our detailed study of general boundary conditions in the supplementary material (Secs. F (3)-(5) and also Sec. C (5).).

Moreover, as the referee has mentioned, there is a topological origin in this phenomena. At the topological phase transition, the change in the topological winding number occurs by changes in the pole structure of the GBZ. This change in the pole structure occur when the poles (bound states) intersect the contour of the GBZ. The intersection of the two-different momenta, by definition, indicates the eigenstate coalescence. As a result, it appears as the exceptional point.

Comment 4

“Minor comment: What is {\mathcal L}_1 and {\mathcal L}_2? I believe they are not defined in the main text.”

Response to comment 4

We thank for point out this. In our manuscript, we have elaborated the definition in more detail the second paragraph on page 3. It now reads as,

“The bulk states $|\psi\rangle$ are characterized by the collections of all generalized momenta (z_1, z_2) with $z_1 z_2 = t_R/t_L$ which form contours in the complex plane that are called GBZs $L_1 = \{z_1\}$, $L_2 = \{z_2\}$. Hence, the contour of different momenta constitutes GBZs $L_1 \times L_2$ due to the constraint $z_1 z_2 = t_R/t_L$.”

Response to Reviewer #2

Introductory comment

"In the manuscript "Topological Phase Transitions of Generalized Brillouin Zone", the authors proposed the general boundary conditions in non-Hermitian systems by adjusting the onsite and hopping strengths of the sample. With the help of the GBC, they demonstrated that a topological state is obtained. The topological states are accompanied by the appearance of some bound states. By adopting the winding number approach, they further point out that the topological phases transition can be captured by the variation of the Generalized Brillouin Zone. Finally, they extended their results to the Kuramoto model. In my opinion, their results are new and valid. The relationships between the proposed GBZ and the existence of the bound states looks rather interesting. However, I have several questions about the topological natures and the definitions of GBZ."

<u>Response to introductory comment</u>

We would like to appreciate the referee for finding our work new and valid. In this response letter, we have addressed the questions accordingly.

Comment 1

"(1)The GBZ proposed in this manuscript seems totally different from the widely used ones. Generally, the GBZ is usually adopted to rebuild the bulk-boundary correspondences for samples under the open and periodic boundary conditions. The GBZ adopted in this manuscript is aimed to rewrite the eigenvalues in the "Hermitian" formula, i.e. $\mathrm{E} \sim \cos(\mathrm{z})$ *instead of* $\mathrm{E} \sim \mathrm{t_L} e^{ik} + t_R e^{ik}$ *for the Hatano-Nelson model. Thus, the definition of the "GBZ" in this manuscript should be clarified more clearly in the main text."*

<u>Response to comment 1</u>

We thank the referee for pointing out an important issue. Indeed, our definition is different from several manuscript, while there are also literatures with the similar definitions. Unlike other works, we use the generalized Brillouin zone (GBZ) using the complex momenta z_1 and z_2 to explain the bulk-boundary correspondence in the general boundary conditions beyond PBC and OBC. In our work, GBZ always has two momenta z_1, z_2, where the wave function is written as,

$$\psi(n) = c_1 z_1^n + c_2 z_2^n.$$

In previous literatures, the two GBZs formed by z_1 and z_2 are considered as equivalent as they only consider in PBC and OBC. For example, in the PBC, the GBZs corresponds to two circles with $|z_1| = 1$, and $|z_2| = t_R/t_L$, respectively. In the OBC, the two GBZs also coincide as $|z_1| = |z_2| = t_R/t_L$, where $t_{R/L}$ is the right and left directional hopping. However, in GBC, we have shown that the two GBZs do not necessarily match each other. Therefore, the two momenta should be considered separately.

For the concreteness, we have also appended the rigorous definitions of the GBZs and corresponding eigenvalue spectra for different boundary conditions in Sec. C of the supplementary material in detail. In our manuscript, we have also elaborated the definition in more detail the first and second paragraph on page 3.

Comment 2

"(2)A topological systems should be robust against some-kinds of global disorders since topology is always regarded as a global concept. Taking the Hatano-Nelson model with Anderson disorders as an example (considering the on-site disorder: $\epsilon_i c_i^\dagger c_i$ *with* $\epsilon_i \in [-\frac{W}{2},\frac{W}{2}]$*), the NHSE and the related winding number still works. The robustness of the NHSE can be clearly identified as well. Thus, the NHSE is considered as the topological feature of non-Hermitian systems.*

However, the definition of the GBC seems to rely on the translational symmetry. With the help of the translational symmetry, one is able to distinct the bulk and the boundary of the sample clearly. However, since the translational symmetry is destroyed everywhere after considering the global disorder [i.e. $\epsilon_i c_i^\dagger c_i$*], the boundary could not be clearly defined. Based on the definition in the main text, every sites gives their GBC. So, does the topological natures still hold? The author should prove that the existence of the bound states are not originate from the local perturbation of the GBC."*

<u>Response to comment 2</u>

We agree with the Referee that in the presence of global disorder, the translational symmetry is destroyed, accordingly, the concept of GBZs are not rigorously applicable. However, we point out that there is a robust topological origin of the bound state in the GBC. The bound state is originated from non-trivial winding number of the boundary matrix $h_B^{+}(z_1,z_2) \equiv \frac{c_1}{c_2}$. As we elaborated in Section E in the supplementary material, the boundary matrix can be physically considered as a reflection amplitude of the scattering matrix. The pole structure (winding number) of the boundary matrix can be only changed by closing the point gap in the spectra. As a result, the boundary states survive up to a critical disorder where the point gap closes. This reasoning is equivalent to the topological stability of the aforementioned NHSE. Although a disorder breaks the translational symmetry, the bound states can remain robust due to this topological origin.

Comment 3

"(3)Some quantities are not clearly defined in the main text, i.e. the quantity "D=0.15" in the Fig. 4."

<u>Response to comment 3</u>

We thank the reviewer for pointing out this. We have defined it in the modified manuscript. It now reads as,

"[See the time evolution of phases and normalized chiral velocity in the presence of random white noise of strength *D* in the} Fig. 4]"

Comment 4

"(4)I am a little bit confused about the meaning to pointing out the eigenstates ($\chi_{L/R}$) and eigenvalues (E) of the Eq. (4). If some quantities are pointed out, they should be adopted to clarify or demonstrate something."

<u>Response to comment 4</u>

We have clarified the connection in the modified manuscript. It now reads as,

"The chiral symmetry of the effective boundary matrix is responsible for the quantization of the non-Bloch topological invariant in the topologically non-trivial phase which we discuss as follows."

Response to Reviewer #3

Introductory comment

"The present work investigates the intrinsic topology of the generalized Brillouin zone. It is shown that the topological boundary modes of the Hatano-Nelson model with the generalized boundary conditions can detect the topological phase transition in, where their wavenumber touche the generalized Brillouin zone and the exceptional points appear. The authors demonstrate that the nonreciprocal Kuramoto model exhibits the topological phase transition by showing the correspondence of the Hatano-Nelson model and the nonreciprocal Kuramoto model. They also propose that the phase transition accompanied by the appearance of the exceptional points results in the exceptional phase transition.

I think that the results obtained in this work are very interesting, because the topology of the single-particle description connects the phenomenon of many-body systems. Meanwhile, it seems that the descriptions of the terminologies and figures are lack. Hence, it would be good to elaborate on unclear points that I denote in the following, by adding the contents of the Method and Supplementary Material to the main text. If the authors appropriately address my points, I would like to recommend publishing the manuscript for Communications Physics."

Response to introductory comment

We thank the referee for finding our work interesting. Below, we answer the referee's questions and included additional information in the main text.

Comment 1

"(i) It is unclear how to read the figures in Fig. 2 because there are no explanations of Fig. 2 in the main text. For instance, what is the meaning of the vector plots in Figs. 2(a)-(c)? What is the difference between Fig. 2(a) and Fig. 2(b)? Where is the exceptional points in Fig. 2(b)? What is the definition of the phase rigidity?"

Response to comment 1

We thank the referee for pointing out this. We have explained Fig. 2 in the main text (page 3 and page 4) and also modified the Fig. 2 for more clarity. It now reads as,

"Fig. 2 shows the topological and exceptional transition of GBZs in the GBC-(I) which we explain in the following manner. In Fig. 2 (a) there are equal number of poles and zeros inside the GBZs and hence the $W_{\mathrm{non-Bloch}}$ vanishes which describes the topologically trivial phase. In Fig. 2 (b) non-zero zero and pole merge with GBZs which describes the exceptional point. In Fig. 2 (c), there are a pair of poles inside GBZs and hence the $W_{\mathrm{non-Bloch}}=1$ which characterizes the topologically non-trivial phase. The merging of the two generalized momenta manifests as the EP, which is signatured by the coalescence of distinct eigenstates and is demonstrated by the vanishing of phase rigidity which is defined as follows

$$R_i = \frac{\langle \psi_i^R | \psi_i^L \rangle}{\langle \psi_i^R | \psi_i^R \rangle}, i \in \{1, 2, \dots, N\}$$

Where $\left|\psi_{\mathrm{i}}^{L(R)}\right\rangle$ are the left (right) eigenmodes corresponding to the eigenvalues E_i. We find that the change in the non-Bloch topological invariant characterizes the number of topological boundary states."

Comment 2

"(ii) I do not know that it is appropriate whether the discrete modes in Fig. 2(c) are called "topological boundary mode" or not. Conventionally, topological boundary modes appear in finite-size system because of the bulk-boundary correspondence. In contrast, the discrete modes appear in the system in the thermodynamic limit, as I refer to the Method. Thus, is there a conventional bulk-boundary correspondence in the topological phase predicted here?"

Response to comment 2

We would like to clarify that there is a discrepancy with the conventional bulk-boundary correspondence, where the boundary spectrum in OBC (finite-size) comes from topology of PBC (thermodynamic limit). In our case, the boundary spectra occur in the given generalized boundary condition due to the topology of the bulk states in the same boundary condition. Although this is different phenomenology from the conventional system, we still refer it to as bulk-boundary correspondence since the emergence of the boundary state is explained by non-trivial topology of macroscopic bulk states. This unusual bulk-boundary correspondence is one of the novelties we have uncovered in our work.

Comment 3

"(iii) I would like to ask the authors to clarify the role of the phase difference between the nearest-neighbor oscillators in the Kuramoto model because I do not know the detail of the model. Why are the boundary conditions determined by the phase difference?"

Response to comment 3

While the Jacobian determines the dynamical stability of the collective motion, its matrix elements are determined by the instant angular coordinate θ_i due to the non-linearity of the system. This is different from single-particle systems, where the Hamiltonian is invariant regardless of the coordinates of the system.

In the Kuramoto model, the interaction between the oscillator is given the sinusoidal interaction as,

$$\frac{d\theta_j}{dt} = \sum_j K_{ij}\sin(\theta_i - \theta_j).$$

Correspondingly, the Jacobian (potential landscape) is given as $J_{ij} = \frac{\partial \frac{d\theta_j}{dt}}{\partial\theta_i} = K_{ij}\cos(\theta_i - \theta_j)$. When $\theta_i - \theta_j = \frac{\pi}{2}, \frac{3\pi}{2}$, the mutual force between the two oscillators become maximal where the potential becomes flat. The Jacobian between the two oscillators vanishes as a result it realizes the effective open boundary condition.

Comment 4

"(iv) It is unclear how to describe the limit-cycle phase in the Kuramoto model. It might be good to clarify the connection between this phase and the Jacobian. Furthermore, what is the definition of \partial_{i} in the Jacobian?"

Response to comment 4

Non-reciprocally coupled Kuramoto model has stable limit cycle phase, known as chiral phase. Each limit-cycle phase is characterized by the topological winding numbers $w = \frac{\sum_i \theta_{i+1} - \theta_i}{2\pi} \in \mathbb{Z}$. In other words, the limit cycle corresponds to the stable motion that has balanced constant phase difference between the nearest-neighbor oscillators as,

$$\theta_{i+1} - \theta_i = \frac{2\pi\omega}{N}.$$

The Jacobian describes the linear excitation around these chiral phases and characterize their local stability. For each instance of the phases, the Jacobian can be written as, $J_{ij} = -\frac{\partial(\partial_t\theta_j)}{\partial\theta_i}$, which emulates the linearized non-interacting Hatano-Nelson model. ∂_i corresponds to the partial derivative in terms of the angular variable in the definition of the Jacobian.

We have now appended more explanations in the main text. It now reads as,

"The Jacobian of the limit cycle phases characterize their local stability and emulates the Hatano-Nelson model with the PBC [Fig. 3 (a)] where the nearest neighbor hopping is determined by the non-reciprocal couplings (K_R, K_L) and phase differences $\delta\theta_{i+1,i} = 2\pi\omega/N$."

Comment 5

"(v) I would like to the authors to add more explanation of the phase slip."

Response to comment 5

The transition between two different chiral phases (say $w \in \mathbb{Z}$ and $w + 1 \in \mathbb{Z}$) can occur by changing the phase difference between the two ends from $2\pi w$ to $2\pi(w+1)$. This phase change process is unstable in dynamical perspective and occur rapidly. Therefore, it is referred to as phase slip. For the general audience, we have added a more explanation, it now reads as,

"The phase transitions between the limit cycles with different phase gradients (say ω and ω') are only possible when one of the phase differences undergoes a change of $2\pi(\omega-\omega')$. This relatively rapid change of the phase difference by $2\pi(\omega-\omega')$ is called *phase slip* where the phase slipping sites can be considered as the boundary. In the interval of the phase slip, the phase difference at the boundary $\delta\theta_{1,N}\equiv\delta\theta_N=\theta_1-\theta_N$ changes to $\delta\theta_{1,N}+2\pi$, and accordingly the boundary hopping terms $J_{1,N}=K_L\cos(\delta\theta_N)$, $J_{N,1}=K_R\cos(\delta\theta_N)$ modify. Therefore, in the interval of the phase slip, the Jacobian (Eq. (8)) emulates the Hatano-Nelson model in the generalized boundary conditions."

Comment 6

"(vi) I think that the occurrence of the limit-cycle instability after the topological phase transition is interesting. To attract readers' attention, it would be good to visually clarify the formation of the positive real eigenvalue from the topological boundary state."

Response to comment 6

We thank for the suggestion. We have newly added the variation of the eigenvalue spectra in the interval of the phase slip in the modified manuscript in Fig. 3 and modified its caption accordingly. We have also added few lines in explanation of Fig. 3 in the third paragraph at page 5 which is read as follows:

"The adiabatic evolution of GBZs and corresponding eigenvalue spectra are shown in Figs. 3 (a)-(d) in the half interval of the phase slip in the top and middle panels of Fig. 3 (since the evolution is symmetric in the other half). The bottom panel of Fig. 3 shows the variation of the eigenvalue spectra and phase rigidity in the interval of the phase slip."

Comment 7

"(vii) What is the meaning of the statement, "The physical consequence is the dynamical restoration of the spontaneously broken U(1) rotation symmetry"?"

Response to comment 7

In the Hermitian Kuramoto model, the synchronization phase corresponds to the stationary motion (when averaged frequency vanishes) $\theta_i(t)\rightarrow\theta_i(0)$ where all the oscillators converges to a specific angle by breaking U(1)-rotation symmetry. However, in our work, we show that non-Hermitian Kuramoto model, the non-reciprocity produces a finite drift motion [Eq. (8) in the main text] that restores U(1)-rotation symmetry.

For the general audience, we appended more detailed explanations. It now reads as, "In other words, in the Hermitian Kuramoto model, the synchronization phase corresponds to the stationary motion (when averaged frequency vanishes) $\theta_i(t) \to \theta_i(0)$ where all the oscillators converge to a specific angle by breaking U(1)-rotation symmetry. However, in the case of the non-Hermitian Kuramoto model, the non-reciprocity produces a finite drift motion that restores U(1)-rotation symmetry."

Comment 8

"(viii) I do not know whether it is appropriate that Eq. (8) characterize the topological phase transition, or not. Does this mean that Eq. (8) is a topological invariant? If so, what is the difference between this quantity and the winding number proposed here?"

<u>Response to comment 8</u>

As the referee has mentioned, Eq. (9) (previously Eq. (8)) characterizes the topological phase transition. We propose Eq. (9) which is the average total velocity of the oscillator, as the macroscopic observable that can captures the topological invariant in the many-body systems. While the winding number is an abstract quantity that cannot be extracted from the collective motion, Eq. (9) is directly related as the macroscopic observable.

Reviewers' comments:

Reviewer #1 (Remarks to the Author):

I am happy about all the responses from the authors. I recommend this manuscript for publication in Communications Physics.

Reviewer #2 (Remarks to the Author):

I have read the reply of the authors to my previous report. The authors adequately address most of my questions. Unfortunately, I do not think the second point of my question is carefully considered by the authors. Instead of some arguments, the numerical or analytical evidence is required to demonstrate the topological natures of the bound states by considering the global disorder. I will require the publication of this manuscript if the above question is clearly identified.

Reviewer #3 (Remarks to the Author):

I thank the authors to answer my questions and clarify my concerns by revising the manuscript. The revised manuscript would satisfy the publication criteria of this journal. Therefore, I would like to recommend publishing the paper for Communications Physics.

Referee response for COMMSPHYS-23-0686-T

We would like to thank the reviewers for carefully reading of our response. We are pleased that two of the reviewers recommended our work for the publication in the *Communications Physics*. In this response letter, we have explicitly addressed Reviewer #2's comments with the additional numerical simulations. Based on the recommendations of two reviewers and response to Reviewer #2, we hope that the paper can be formally accepted for the publication in *Communications Physics*.

Response to Reviewer #1

General Comment

"I am happy about all the responses from the authors. I recommend this manuscript for publication in Communications Physics."

Response to comment

We appreciate the reviewer for the recommendation of the publication. In addition, we also thank for giving comments to improve our manuscript.

Response to Reviewer #3

General Comment

"I thank the authors to answer my questions and clarify my concerns by revising the manuscript. The revised manuscript would satisfy the publication criteria of this journal. Therefore, I would like to recommend publishing the paper for Communications Physics."

Response to comment

We are happy that the reviewer recommends our manuscript for the publication. We also appreciate for giving helpful comments to improve our manuscript.

Response to Reviewer #2

Comment

"I have read the reply of the authors to my previous report. The authors adequately address most of my questions. Unfortunately, I do not think the second point of my question is carefully considered by the authors. Instead of some arguments, the numerical or analytical evidence is required to demonstrate the topological natures of the bound states by considering the global disorder. I will require the publication of this manuscript if the above question is clearly identified."

Response to comment

In this response, we have appended the additional numerical calculation that considers the random disorder. We show the topological origin of the bound states by showing that,

(I) The continuous manifold of the generalized Brillouin Zone (GBZ) is still well-defined in the presence of the moderate strength of the disorder. Accordingly, the topological invariant is well-defined even without the translational symmetry.

(II) A numerical Fourier transformed spectra of the eigenstates shows the robust boundary states in the complex momentum plane, which is well-isolated from the bulk states by the point gap.

Using two methods, the topological nature of the bound states still survives for moderate disorder strength. In the strong disorder case, where the disorder strength is comparable to the size of the point gap, the topological property is lost like other topological insulator phases.

To start the discussion, we consider the Hamiltonian in the presence of the random disorder, which can be written as,

$$H = \sum_{i=1}^{N-1} \left[t_R c_{i+1}^\dagger c_i + t_L c_i^\dagger c_{i+1} \right] + t_R' c_1^\dagger c_N + t_L' c_N^\dagger c_1 + \epsilon_1 c_1^\dagger c_1 + \epsilon_N c_N^\dagger c_N + \sum_{i=1}^{N} w_i\, c_i^\dagger c_i,$$

where $w_i \in [-w/2, w/2]$ is a uniformly distributed random number representing the onsite potential at the *i*-th site. The eigenstates that $H|\Psi_n\rangle = E_n|\Psi_n\rangle$ satisfy the bulk difference equation, which is given as,

$$t_R \psi_{i-1} + (w_i - E_n)\psi_i + t_L \psi_{i+1} = 0,$$

The above equations generally have two roots of generalized momenta z_1, z_2. In addition, the pair of the generalized momenta satisfies the relation,

$$z_1 z_2 = t_R / t_L,$$

regardless of the disorder potential strength. As a result, one can define the pair of the continuous manifolds of the bulk GBZ for each site *i*.

The GBZs are separately defined for each site since the bulk equation is dependent on the disorder strength at the site-i. For sufficiently small disorder strength, the continuous manifold of the GBZ and the subsequent winding number is well-defined [See Fig. 2]. In addition, we find the well-separated momenta of the boundary states of the bulk GBZ. As the disorder strength increases, where the disorder strength reaches the scale of the point gap, the complex momenta of the boundary states merge with those of the bulk states. In this case, the topological feature is lost [See Fig. 3]. We find that the critical disorder strength, at which the bound states lose their topological nature, occurs when the inverse participation ratio (IPR) and phase rigidity (PR) of the bound state decreases and merges with those of the bulk states [See Fig. 3].

We strengthen our argument by explicitly performing the numerical Fourier transform to calculate the complex momenta spectra. In the complex momenta, the real part characterizes the wave vector, and the imaginary part characterizes the damping or amplification. For each numerically obtained eigenstates, we can define the projection operator at every point of the complex plane as follows:

$$P(z) = |\Psi(z)\rangle\langle\Psi(z)|, \text{ where } |\Psi(z)\rangle = (z\, z^2 \ldots z^N)^T.$$

The eigenstates are generally linear combinations of different generalized momenta $|\Psi_n\rangle = (\psi_1 \ldots \psi_N)^T, \psi_i = c_1 z_1^i + c_2 z_2^i$ (at least for zero disorder strength). Then, the projection amplitude $S(z, n) = |\langle\Psi(z)|\Psi_n\rangle|^2$ has a peak around the dominant generalized momenta. The broadening of the peak is given by the imaginary part of the momenta. In the clean limit [Fig. 1(1-c)], we find the ring-shaped contour of the peaks, which indicates the bulk GBZ. In addition, we find the spectra of the pair of the boundary states inside and outside the GBZ. As the moderate strength of the disorder is considered [Fig. 1(2-c)], the peaks broaden but still one can distinguish the peak of the bound state from the bulk states. Therefore, the topological nature of the bound states still survives. For sufficiently large disorder strengths [Fig. 1(3-c)], the distinction between the peaks is lost and the bound state loses its topological nature.

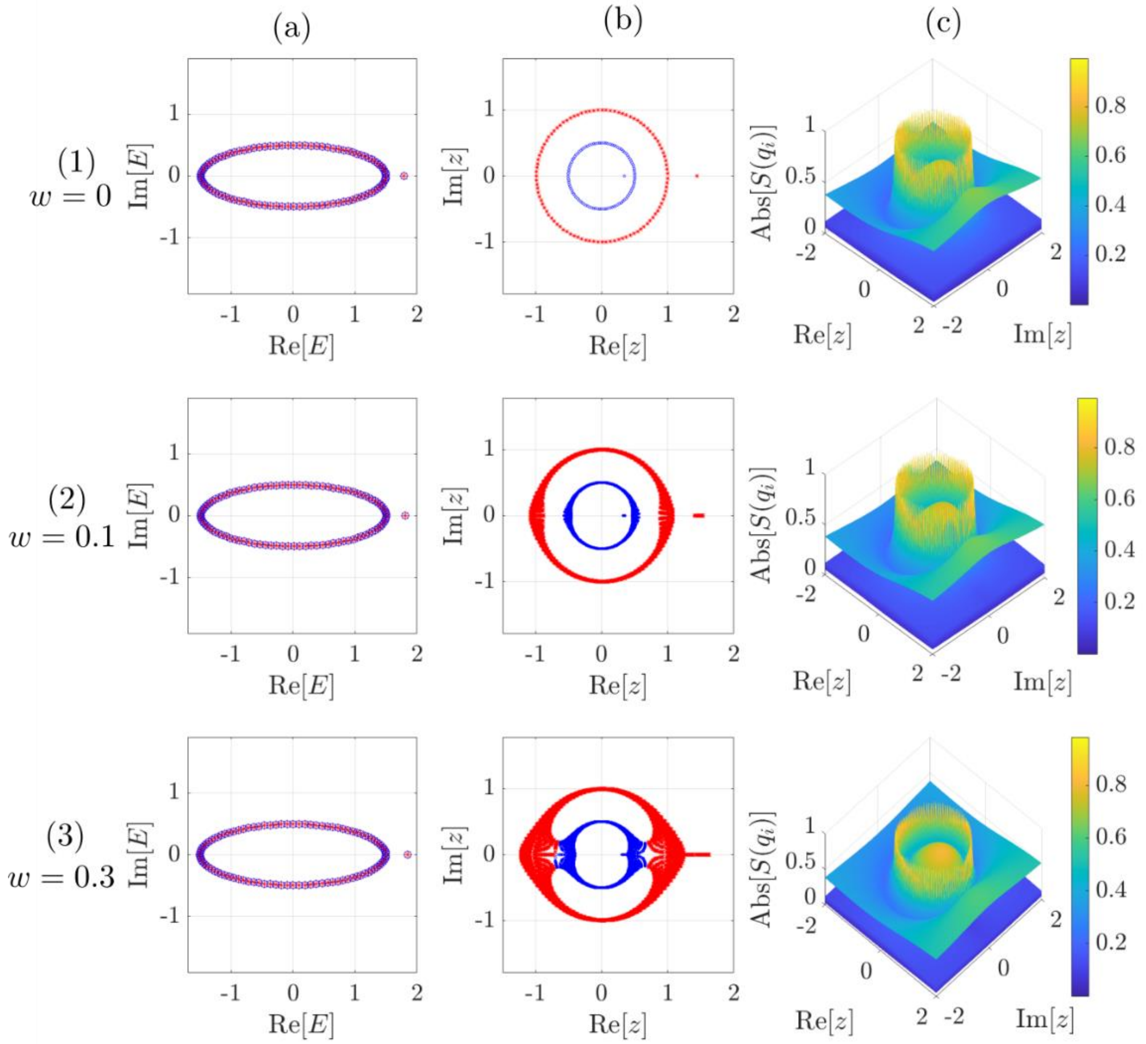

Fig. 1: (a) Eigenvalue spectra, (b) GBZs $\mathcal{L}^{(i)}$, for $i \in [2, N-1]$ using finite difference method, and (c) GBZs using wave function projection method for different values of disorder strengths $w = 0$[panel (1)], $w = 0.1$ [panel (2)], $w = 0.3$ [panel (3)]. Parameters: $N = 100, t_L = t_L^{'} = 1, t_R = t_R^{'} = 0.5, \epsilon_1 = 1.1, \epsilon_N = 0$ are used

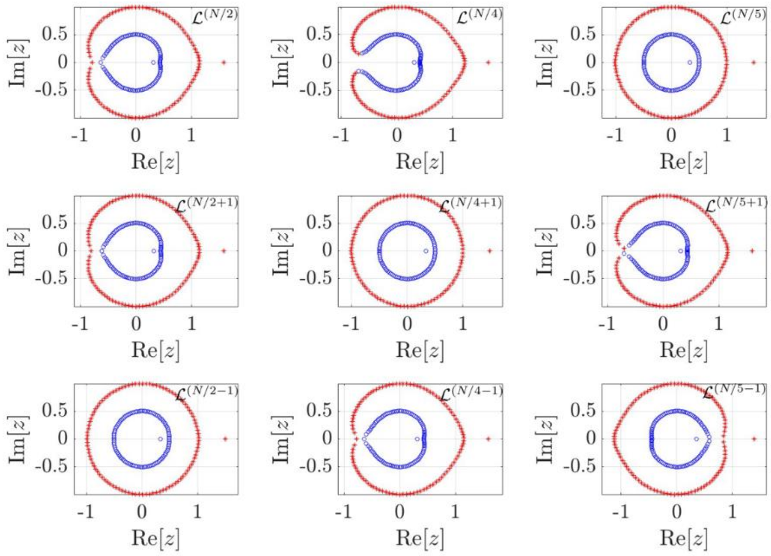

Fig. 2: Collections of complex generalized momenta ($\mathcal{L}^{(i)}$) for i −th site in the presence of disorder of strength $w = 0.3$. Parameters: $N = 100, t_L = t_L^{'} = 1, t_R = t_R^{'} = 0.5, \epsilon_1 = 1.1, \epsilon_N = 0$.

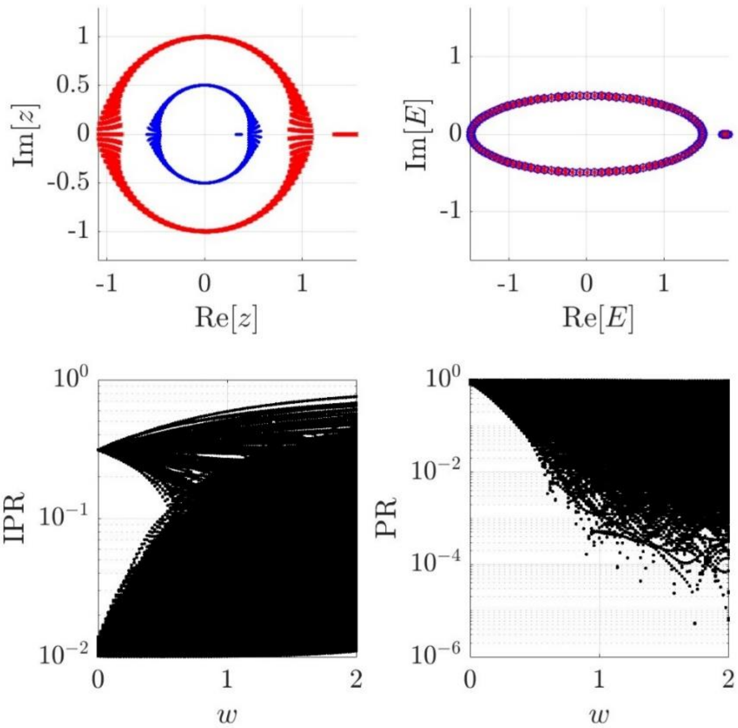

Fig. 3: Top panel: GBZs (left) and eigenvalue spectra (right) in the presence of global disorder of strength $w = 0.3$ for 100 disorder realizations. Bottom panel: individual inverse participation ratio (left) and phase rigidity (right) as a function of disorder strength w for 100 disorder realization. Parameters: $N = 100, t_L = t_L^{'} = 1, t_R = t_R^{'} = 0.5, \epsilon_1 = 1.1, \epsilon_N = 0$.

REVIEWERS' COMMENTS:

Reviewer #2 (Remarks to the Author):

I think the disorder effect is cearfully considered by the authors. The topological nature of the proposed structure can be clearly identified even the disorder exists. I would like to recommend publishing the paper for Communications Physics.