nature portfolio

Peer Review File

Time-dependent Gutzwiller simulation of Floquet topological superconductivity

Reviewers' comments:

Reviewer #1 (Remarks to the Author):

There has been steady interest in topological materials, including topological superconductors, for over a decade. Interest in topological superconductors is strongly linked to the possibility of observing Majorana fermions, yet achieving topological superconductivity has been challenging in the lab. This fact has spurred a substantial effort in generating topological superconductivity artificially by driving an ordinary superconductor using electromagnetic radiation – these are the so-called Floquet topological superconductors, which exist as non-equilibrium phases. In light of this, considerable attention has been devoted to superconductors' responses to radiation, which, however, remains incompletely understood. In this context the paper by Anan, Morimoto and Kitamura is timely and considers the problem of a d-wave superconductor irradiated by circularly polarized light and the emergence of a phase with nonzero Chern number in response to radiation. The authors use the Gutzwiller projection to reduce double occupancy and study the Floquet t-J model in the mean-field approximation, which yields an approximate time evolution for the superconducting many-body state, mimicking a cuprate superconductor. The methodology is sound and the presentation is clear, while the results will be useful to the community at large in understanding the properties of this elusive state of matter. The paper can be considered for publication once the following points are resolved.

1. The effective Hamiltonian Eq. 6 is based on the Schrieffer-Wolff (SW) transformation. I understand that a more complete description is available in Refs. 28-29, but for the presentation to be consistent the authors need to specify why SW is justified here: what are the small parameters, what would happen if terms of third order in SW are considered, and when does the approximation break down? In particular, can third-order terms interfere with the topological phases found under the action of CPL?

2. As expected on physical grounds, \Delta d_xy becomes nonzero under the action of CPL, which leads to topological superconductivity. For the latter it is obvious that the size of d_xy matters. The authors study this size as a function of E and \omega. To what extent is the size of \Delta d_xy dependent on the parameters chosen for the simulation (t, \delta, U, etc)?

3. Similarly, could the authors explain the role of the divergent terms mentioned in connection to Eqs. 18 and 19? I understand that these terms give rise to divergences, but why are these divergences simply removed rather than being regularized?

4. Finally, although beyond the scope of this paper, the following is important for practical applications. I appreciate the fact that the authors discuss the application of their theory to practical materials. In this context the authors find the topological gap to be 0.03 x the d-wave superconducting gap, and use a sample T_c for cuprates to estimate the topological gap. To what extent is this result affected by disorder? 3K is not an especially large number hence some comments are in order on whether we should still expect such a gap when disorder, inevitable in experimental samples, is present, and which types of disorder would be the most harmful.

Reviewer #2 (Remarks to the Author):

In their article, the Authors explore the possibility to induce topological superconductivity out of non-topological superconductivity, by applying time dependent electric fields. Remarkably, the required laser intensity is smaller than in previous similar studies.
I find the results sound, interesting and timely. Furthermore the article is extremely well written, clear, and complete, despite the complexity of the theoretical treatments.
I hence strongly support its acceptance almost in the present form.
I would just ask for the addition of a clear statement about the range of validity of the analysis below

Eq.3, and for the correction of the typo 'Several experimental methods is considered'. Optionally, I think that the addition of a 'hand-waving' argument for the obtained results, if available, would definitly help inspiring research from other groups on the topic.

Reviewer #3 (Remarks to the Author):

The present manuscript deals with the theoretical suggestion of topological superconductivity in the presence of many-body effects upon irradiation of circularly polarised light. The Authors consider the Hubbard model with nearest and next-nearest neighbours hopping. By means of a unitary transformation they restrict the analysis in the lowest charge-diagonal sector and use the Gutzwiller wave function and the Gutzwiller approximation to evaluate the effective action. Via standard BCS decoupling they obtain an effective BdG time-dependent (due to the effect of the radiation field) equation. The time evolution of the ensuing solution is analysed in the context of the Floquet band-theory, by evaluating the Berry curvature and Chern numbers as signature of the topological transition.
The method used allows to explore both high and low irradiation frequency regimes and this provides an advantage with respect to other methods.

The paper is very well written and the results are scientifically sound and presented in a convincing way. I think that this paper, if published, will provide a valuable contribution to the field and I advise publication. Before that, however, I feel that the paper may further be improved if the Authors consider, and comment, the following points:

Is there a specific advantage and/or scientific merit in adopting the Gutzwiller approximation as compared to other well-known techniques for strongly correlated systems such as the dynamical mean-field theory and/or the slave-boson approach?
Why the Authors choose a specific doping level (delta =0.2)? This is because the superconductivity is only relevant for that value? How computationally demanding would be to analyse the model for other values of the doping level?

Reviewers' comments:
Reviewer #1 (Remarks to the Author):
Remarks from Reviewer #1-0

There has been steady interest in topological materials, including topological superconductors, for over a decade. Interest in topological superconductors is strongly linked to the possibility of observing Majorana fermions, yet achieving topological superconductivity has been challenging in the lab. This fact has spurred a substantial effort in generating topological superconductivity artificially by driving an ordinary superconductor using electromagnetic radiation – these are the so-called Floquet topological superconductors, which exist as non-equilibrium phases. In light of this, considerable attention has been devoted to superconductors' responses to radiation, which, however, remains incompletely understood. In this context the paper by Anan, Morimoto and Kitamura is timely and considers the problem of a d-wave superconductor irradiated by circularly polarized light and the emergence of a phase with nonzero Chern number in response to radiation. The authors use the Gutzwiller projection to reduce double occupancy and study the Floquet t-J model in the mean-field approximation, which yields an approximate time evolution for the superconducting many-body state, mimicking a cuprate superconductor. The methodology is sound and the presentation is clear, while the results will be useful to the community at large in understanding the properties of this elusive state of matter. The paper can be considered for publication once the following points are resolved.

Reply to Reviewer #1-0

First of all, we would like to thank the reviewer for appreciating the scientific value of this paper and for providing constructive questions and comments (as itemized below), all of which are useful to make the present paper more convincing.

Remarks from Reviewer #1-1

The effective Hamiltonian Eq. 6 is based on the Schrieffer-Wolff (SW) transformation. I understand that a more complete description is available in Refs. 28-29, but for the presentation to be consistent the authors need to specify why SW is justified here: what are the small parameters, what would happen if terms of third order in SW are considered, and when does the approximation break down? In particular, can third-order terms interfere with the topological phases found under the action of CPL?

Reply to Reviewer #1-1

We thank the reviewer for pointing out the need for a clearer description of the justification for using the Schrieffer-Wolff (SW) transformation. We use the time-periodic SW transformation that derives *t–J* model in the off-resonant condition. It is justified when $t_{ij}^{(m)}$ are sufficiently smaller than $U - n\omega$ in Eqs. (7) and (9), therefore SW transformation breaks down at $t_{ij}^{(m)} \sim U - n\omega$.

Therefore, we add the statement about the range of validity of SW transformation in the subsection "Formalism " in Results as following: "The time-periodic Schrieffer-Wolff expansion is a perturbative expansion in condition $t_0 \ll U$ meanwhile there is no resonance between ω and U (i.e. $t_{ij}^{(m)} \ll U - n\omega$)."

Even if higher order terms in SW transformation are considered, they do not so much interfere with the topological phase under the CPL because they are smaller than the terms of second-order, in principle. When it comes to the effect of higher order terms in SW transformation, previous studies discuss the effect of fourth-order terms. If terms of fourth-order in SW transformation are considered, scalar spin chirality would be induced [Kitamura et al., PRB 2017]. And this term produces other off-diagonal terms in BdG Hamiltonian inducing topological superconductivity[Kitamura and Aoki, *Commun. Phys.* 2022].

In revised manuscript, we refer the effect of terms of fourth-order in SW transformation in the subsection "Schrieffer-Wolff transformation" in Methods as following: "In the present method, we perform the Schrieffer-Wolff transformation up to the second-order of hopping. This generates three-site term which induces topological superconductivity. Moreover, if terms of fourth-order in Schrieffer-Wolff transformation are considered, scalar spin chirality terms scalar spin chirality terms develop, which also induce topological superconductivity [Kitamura and Aoki, *Commun. Phys.* 2022]."

Remarks from Reviewer #1-2

As expected on physical grounds, Δd_{xy} becomes nonzero under the action of CPL, which leads to topological superconductivity. For the latter it is obvious that the size of d_{xy} matters. The authors study this size as a function of E and ω. To what extent is the size of Δd_{xy} dependent on the parameters chosen for the simulation (t, δ, U, etc)?

Reply to Reviewer #1-2

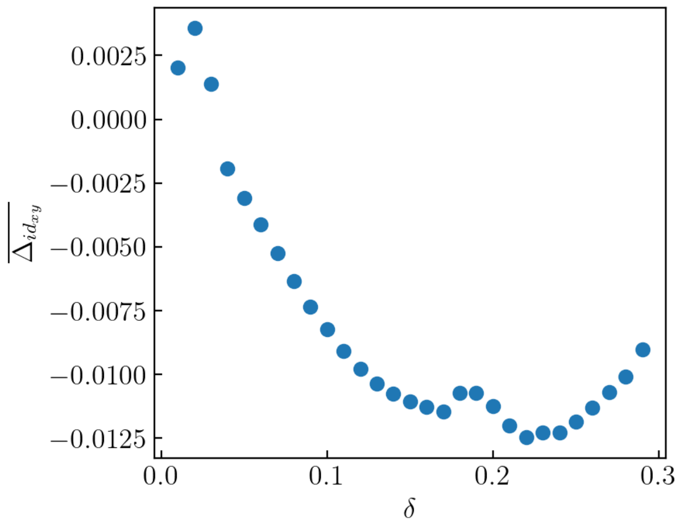

FIG. 1. Doping dependence of $\overline{\Delta_{id_{xy}}}$ in the high-frequency region($\omega = 5.8t_0, E = 5.4t_0/a$)

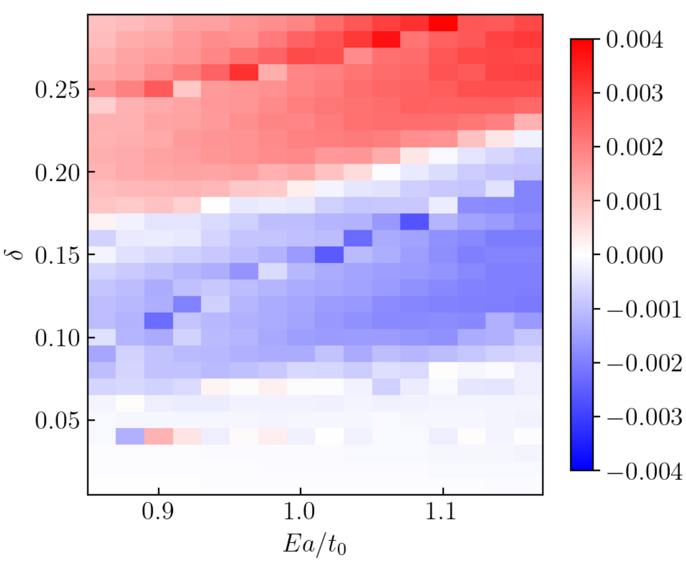

FIG. 2. A color plot of $\overline{\Delta_{id_{xy}}}$ as the function of electric field amplitude E and doping rate δ in the low-frequency region ($\omega = 1.03t_0$)

We thank the reviewer for raising this point. We present the doping dependence of $\overline{\Delta_{id_{xy}}}$ in the high-frequency region($\omega = 5.8t_0, E = 5.4t_0/a$) in Fig 1. The three-site term is proportional to the doping rate δ, whereas in the Gutzwiller approximation, $\Delta_{d_{x^2-y^2}}$ decreases monotonically with δ. Consequently, the absolute value of $\overline{\Delta_{id_{xy}}}$ becomes largest at the intermediate values of δ around $\delta \sim 0.2$.

Next, we show the color plot of $\overline{\Delta_{id_{xy}}}$ as a function of electric field amplitude E and doping rate δ in the low-frequency region ($\omega = 1.03t_0$) in Fig. 2. The electric field amplitude E where the topological phase transition takes place (shown in Fig. 3(d)) varies with the doping rate δ. This is because the three-site term is proportional to the doping rate, as seen from Eq. (12). Additionally, note that the Gutzwiller approximation tends to overestimate the superconducting pairing amplitude for doping

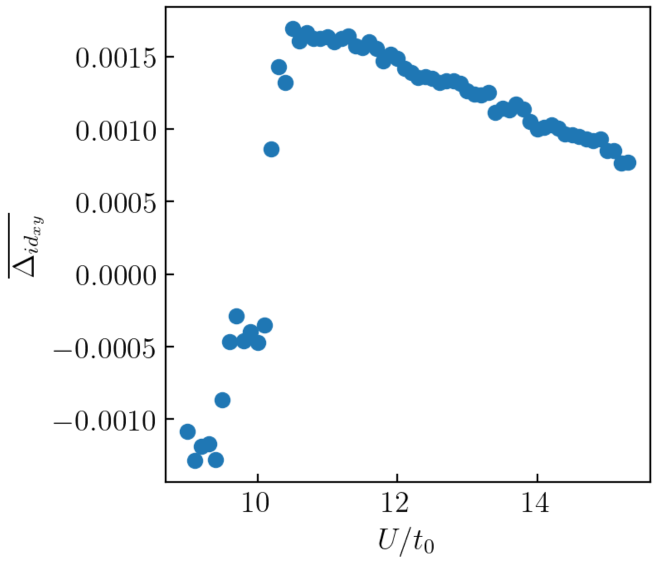

FIG. 3. U/t_0 dependence of $\overline{\Delta_{id_{xy}}}$ in the low-frequency region ($\omega = 1.03t_0, E = 0.96a/t_0$)

levels less than the optimal dope, and hence, this result overestimates $\overline{\Delta_{id_{xy}}}$ for the values of δ lower than 0.2.

Furthermore, we present the dependency of $\overline{\Delta_{id_{xy}}}$ on U/t_0 in the low-frequency region ($\omega = 1.03t_0, E = 0.96a/t_0$) in Fig. 3. A sharp change, along with a change in sign, is observed for $U < 11t_0$ while a more gradual change is observed for $U > 11t_0$. Qualitatively, $\overline{\Delta_{id_{xy}}}$ at this parameter $\omega = 1.03t_0, E = 0.96a/t_0$ mainly stems from the contribution of the $n = 10$ term in Eq. (19), and the change in sign of $U - n\omega$ in Eq. (19) leads to the sharp change for $U < 11t_0$.

In the revised manuscript, we have added Supplementary information about above results and discussion.

Remarks from Reviewer #1-3

Similarly, could the authors explain the role of the divergent terms mentioned in connection to Eqs. 18 and 19? I understand that these terms give rise to divergences, but why are these divergences simply removed rather than being regularized?

Reply to Reviewer #1-3

Divergence in Eqs. (18) and (19) means a breakdown of SW transformation in the off-resonant case. This can be intuitively understood from the denominator in Eqs. (18) and (19) involving $U - n\omega$, which diverges in the case of resonance with the Hubbard interaction. As mentioned in Reply to Reviewer #1-1, the SW transformation is no longer justified around these divergences.

To clarify this point, we revised the sentence in the subsection "Time evolution of superconducting pairing amplitudes" in Results as

"Note that for $\omega = 4t_0,\ 6t_0$, Eqs. (18) and (19) contain divergent terms with the denominator $U - n\omega$ for $U = 12t_0$, with which the effective Hamiltonian becomes ill-defined because of breaking down of the Schrieffer-Wolff transformation."

Remarks from Reviewer #1-4

Finally, although beyond the scope of this paper, the following is important for practical applications. I appreciate the fact that the authors discuss the application of their theory to practical materials. In this context the authors find the topological gap to be 0.03 x the d-wave superconducting gap, and use a sample T_c for cuprates to estimate the topological gap. To what extent is this result affected by disorder? 3K is not an especially large number hence some comments are in order on whether we should still expect such a gap when disorder, inevitable in experimental samples, is present, and which types of disorder would be the most harmful.

Reply to Reviewer #1-4

We thank the reviewer for raising the importance of the effect of disorder. Since the present $d + id$-wave superconductivity is based on the $d_{x^2-y^2}$-wave superconductivity in the undriven cuprates, we consider that the suppression of the topological superconductivity by the impurities will be qualitatively the same as that for the original d-wave superconductivity in cuprate superconductors. If some cuprate sample exhibits a T_c of approximately 90 K, it would likely be sufficient in terms of sample quality to achieve a topological gap on the order of 3 K. As for the types of disorder, in general, both magnetic and non-magnetic impurities suppress superconductivity to the same extent except for s-wave superconductors [Golubov and Mazin, PRB 1997]. Therefore, it is better to avoid both types of disorder for the observation of the topological gap.

In revised manuscript, we add the comment on the affect of the impurities in Discussion as following: "In addition, as for the impurity effect, since the present $d + id$-wave superconductivity is based on the $d_{x^2-y^2}$-wave superconductivity in the undriven cuprates, we consider that the suppression of the topological superconductivity by the impurities will be qualitatively the same as that for the original d-wave superconductivity in cuprate superconductors."

. . . .

Reviewer #2 (Remarks to the Author):

Remarks from Reviewer #2-0

In their article, the Authors explore the possibility to induce topological superconductivity out of non-topological superconductivity, by applying time dependent electric fields. Remarkably, the required laser intensity is smaller than in previous similar studies.

I find the results sound, interesting and timely. Furthermore the article is extremely well written, clear, and complete, despite the complexity of the theoretical treatments. I hence strongly support its acceptance almost in the present form.

Reply to Reviewer #2-0

We would like to thank the reviewer for appreciating the scientific value of our work correctly and recommending the manuscript for publication. Below are the answers for the comments.

Remarks from Reviewer #2-1

I would just ask for the addition of a clear statement about the range of validity of the analysis below Eq.3, and for the correction of the typo 'Several experimental methods is considered'.

Reply to Reviewer #2-1

We thank the referee for pointing the typo. We corrected in the revised manuscript. Also we thank the reviewer for pointing out the need for a clearer description of the justification for using the Schrieffer-Wolff (SW) transformation. We use the time-periodic SW transformation that derives t–J model in the off-resonant condition. It is justified when $t_{ij}^{(m)}$ are sufficiently smaller than $U - n\omega$ in Eqs. (7) and (9), therefore SW transformation breaks down at $t_{ij}^{(m)} \sim U - n\omega$.

Therefore, we add the statement about the range of validity of SW transformation in the subsection "Formalism" in Results as following: "The time-periodic Schrieffer-Wolff expansion is a perturbative expansion in condition $t_0 \ll U$ meanwhile there is no resonance between ω and U (i.e. $t_{ij}^{(m)} \ll U - n\omega$)."

Remarks from Reviewer #2-2

Optionally, I think that the addition of a 'hand-waving' argument for the obtained results, if available, would definitly help inspiring research from other groups on the topic.

Reply to Reviewer #2-2

We thank the reviewer for recommending that we add intuitive explanations for the obtained topological superconductivity. When we consider the spin part of the three-site terms $\sum_{\sigma\sigma'} \hat{c}_{i\sigma}^{\dagger}\boldsymbol{\sigma}_{\sigma\sigma'}\hat{c}_{k\sigma'} \cdot \hat{S}_j$ where $i-j$ is a next-nearest neighbor bond and $j-k$ is a nearest neighbor bond as shown in Fig. 4, the three-site terms generate pairing amplitude on each bond. When the coefficients of the three-site terms $\tilde{\Gamma}(t)$ in Eq. (6) become complex due to the circularly polarized light, a complex phase is introduced to the next-nearest neighbor pairing amplitude. The same applies to the density part of the three-site term $\sum_{\sigma} \hat{c}_{i\sigma}^{\dagger}\hat{c}_{k\sigma}\hat{n}_j$.

Therefore, in the revised manuscript, we have added the explanations for the CPL induced topological superconductivity in the subsection "Formalism" in Results as following: "When we consider the spin part of the three-site terms $\sum_{\sigma\sigma'} \hat{c}_{i\sigma}^{\dagger}\boldsymbol{\sigma}_{\sigma\sigma'}\hat{c}_{k\sigma'} \cdot \hat{\boldsymbol{S}}_j$ where i-j is a next-nearest neighbor bond and j-k is a nearest neighbor bond, the three-site terms generate pairing amplitude

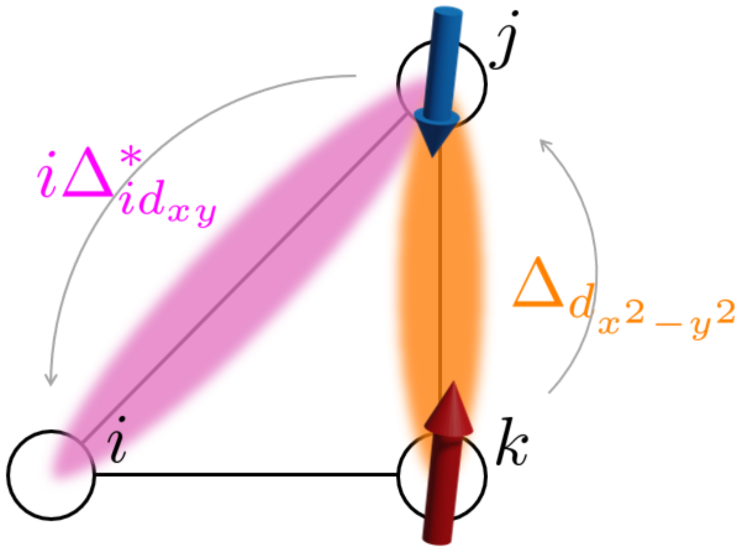

FIG. 4. Schematic picture of $\Delta_{id_{xy}}$ stemming from three-site term

on each bond as $\sum_{\sigma\sigma'} \hat{c}^\dagger_{i\sigma} \boldsymbol{\sigma}_{\sigma\sigma'} \hat{c}_{k\sigma'} \cdot \hat{\boldsymbol{S}}_j \sim \left\langle \hat{c}^\dagger_{i\downarrow} \hat{c}^\dagger_{j\uparrow} \right\rangle \left\langle \hat{c}_{j\uparrow} \hat{c}_{k\downarrow} \right\rangle$. The pairing amplitudes $\left\langle \hat{c}^\dagger_{i\downarrow} \hat{c}^\dagger_{j\uparrow} \right\rangle$ and $\left\langle \hat{c}_{j\uparrow} \hat{c}_{k\downarrow} \right\rangle$ correspond to d_{xy}-wave components and $d_{x^2-y^2}$-wave components of pairing amplitudes respectively. When the coefficients of the three-site terms $\tilde{\Gamma}(t)$ in Eq. (6) become complex due to the CPL, the next-nearest neighbor pairing amplitude $\left\langle \hat{c}^\dagger_{i\downarrow} \hat{c}^\dagger_{j\uparrow} \right\rangle$ becomes complex, embodying the id_{xy}-wave component of the pairing amplitude (as shown in Fig. 4). The same applies to the density part of the three-site term $\sum_\sigma \hat{c}^\dagger_{i\sigma} \hat{c}_{k\sigma} \hat{n}_j$."

. . . .

Reviewer #3 (Remarks to the Author):
Remarks from Reviewer #3-0

The present manuscript deals with the theoretical suggestion of topological superconductivity in the presence of many-body effects upon irradiation of circularly polarised light. The Authors consider the Hubbard model with nearest and next-nearest neighbours hopping. By means of a unitary transformation they restrict the analysis in the lowest charge-diagonal sector and use the Gutzwiller wave function and the Gutzwiller approximation to evaluate the effective action. Via standard BCS decoupling they obtain an effective BdG time-dependent (due to the effect of the radiation field) equation. The time evolution of the ensuing solution is analysed in the context of the Floquet band-theory, by evaluating the Berry curvature and Chern numbers as signature of the topological transition. The method used allows to explore both high and low irradiation frequency regimes and this provides an advantage with respect to other methods.

The paper is very well written and the results are scientifically sound and presented in a convincing way. I think that this paper, if published, will provide a valuable contribution to the field and I advise publication. Before that, however, I feel that the paper may further be improved if the Authors consider, and comment, the following points:

Reply to Reviewer #3-0

We would like to thank the reviewer for appreciating the scientific value of this paper and for providing constructive questions and comments.

Remarks from Reviewer #3-1

Is there a specific advantage and/or scientific merit in adopting the Gutzwiller approximation as compared to other well-known techniques for strongly correlated systems such as the dynamical mean-field theory and/or the slave-boson approach?

Reply to Reviewer #3-1

First, the advantage of using the Gutzwiller approximation over the dynamical mean-field theory (DMFT) lies in its simpler handling. To deal with d-wave superconductivity using DMFT, a self-consistent cluster approximation is necessary rather than the isotropic local impurity self-consistent approximation [Georges et al., *Rev. Mod. Phys.* 1996]. Indeed, there are studies analyzing cuprates using cluster DMFT [Lichtenstein and Katsnelson, PRB 2000;Karp et al., PRB 2022], but it is considered challenging to execute this for every time step to calculate time evolution.

While it is possible to calculate the time evolution of cuprates under circularly polarized light using the slave-boson method, the holon operators, which do not carry a spin index, are diagonal [Ogata et al., *Rep. Prog. Phys.* 2008], and circularly polarized light does not affect the condensation of holon $\left\langle \hat{b}_i \hat{b}_j \right\rangle$ and spinon $\left\langle \hat{f}_{i\uparrow} \hat{f}_{j\downarrow} \right\rangle$, and hence the superconducting pairing amplitude $\left\langle \hat{c}_{i\uparrow} \hat{c}_{j\downarrow} \right\rangle \sim \left\langle \hat{f}_{i\uparrow} \hat{f}_{j\downarrow} \right\rangle \left\langle \hat{b}_i \hat{b}_j \right\rangle$. Therefore, it is expected that using the slave-boson method would yield similar results to those obtained with the Gutzwiller approximation. However, the Gutzwiller approximation tends to overestimate the superconducting pairing amplitude for doping levels less than the optimal, so in such cases, it is considered better to use the slave-boson method.

In the revised manuscript, we have added a paragraph regarding the comparison with the slave-boson method in Supplementary information as folloing: "While we employed the Gutzwiller approximation to simulate the time evolution of cuprates under circularly polarized light, we can alternatively use the slave-boson method instead of the Gutzwiller approximation. In the slave-boson method, the holon operators, which do not carry a spin index, are diagonal [Ogata et al., *Rep. Prog. Phys.* 2008] and circularly polarized light does not affect the condensation of holon $\left\langle \hat{b}_i \hat{b}_j \right\rangle$ and spinon $\left\langle \hat{f}_{i\uparrow} \hat{f}_{j\downarrow} \right\rangle$, and hence the superconducting pairing amplitude $\left\langle \hat{c}_{i\uparrow} \hat{c}_{j\downarrow} \right\rangle \sim \left\langle \hat{f}_{i\uparrow} \hat{f}_{j\downarrow} \right\rangle \left\langle \hat{b}_i \hat{b}_j \right\rangle$. Therefore, it is expected that using the slave-boson method would yield similar results to those obtained with the Gutzwiller approximation. However, the Gutzwiller approximation tends to overestimate the superconducting pairing amplitude for doping levels less than the optimal, so in such cases, it is considered better to use the slave-boson method."

Remarks from Reviewer #3-2

> *Why the Authors choose a specific doping level (delta =0.2)? This is because the superconductivity is only relevant for that value? How computationally demanding would be to analyse the model for other values of the doping level?*

Reply to Reviewer #3-2

In the original manuscript, the doping rate was fixed at 0.2 because it is in the vicinity of the typical optimal doping of cuprates. The computational cost of the present method does not vary with the doping level, and below we show the results in various doping levels.

We present the doping dependence of $\overline{\Delta_{id_{xy}}}$ in the high-frequency region($\omega = 5.8t_0, E = 5.4t_0/a$) in Fig 5. The three-site term is proportional to the doping rate δ, whereas in the Gutzwiller approximation, $\Delta_{d_{x^2-y^2}}$ decreases monotonically with δ. Consequently, the absolute value of $\overline{\Delta_{id_{xy}}}$ becomes largest at the intermediate values of δ around $\delta \sim 0.2$.

Next, we show the color plot of $\overline{\Delta_{id_{xy}}}$ as a function of electric field amplitude E and doping rate δ in the low-frequency region ($\omega = 1.03t_0$) in Fig. 6. The electric field amplitude E where the topological phase transition takes place (shown in Fig. 3(d)) varies with the doping rate δ. This is because the three-site term is proportional to the doping rate, as seen from Eq. (12). Additionally, note that the Gutzwiller approximation tends to overestimate the superconducting pairing amplitude for doping levels less than the optimal dope, and hence, this result overestimates $\overline{\Delta_{id_{xy}}}$ for the values of δ lower than 0.2.

In the revised manuscript, we have added Supplementary information about above results and discussion.

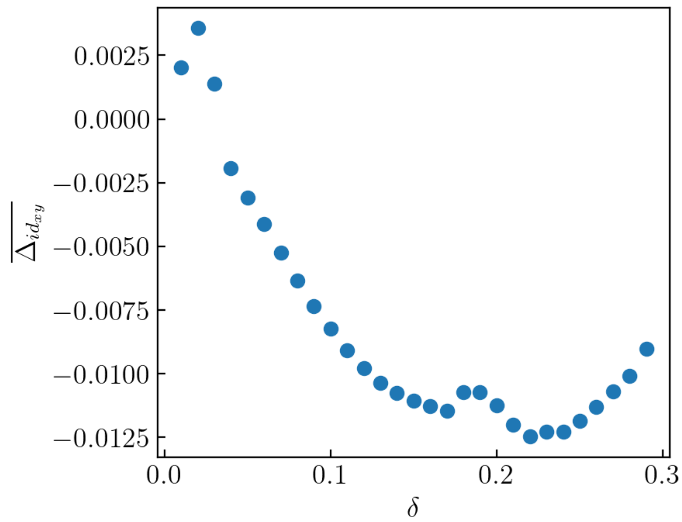

FIG. 5. Doping dependence of $\overline{\Delta_{id_{xy}}}$ in the high-frequency region($\omega = 5.8t_0, E = 5.4t_0/a$)

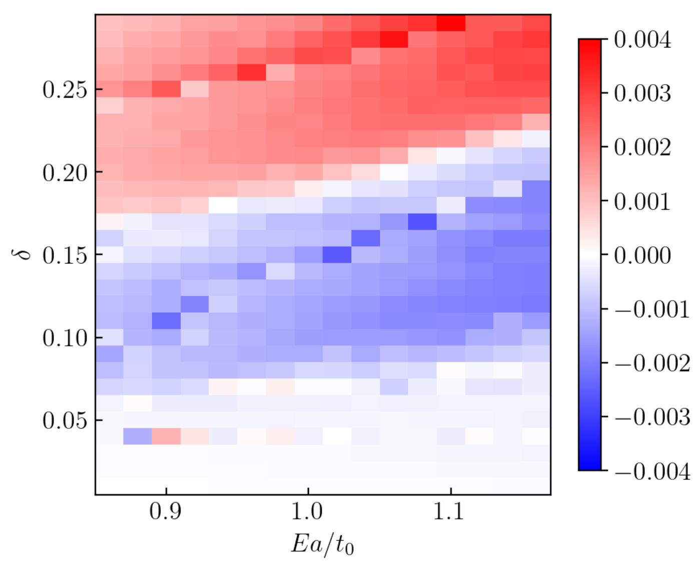

FIG. 6. A color plot of $\overline{\Delta_{id_{xy}}}$ as the function of electric field amplitude E and doping rate δ in the low-frequency region ($\omega = 1.03t_0$)

REVIEWERS' COMMENTS:

Reviewer #1 (Remarks to the Author):

I have examined carefully all the referee reports, the authors' response and the revised manuscript. I note that all referees pointed out the timeliness of the manuscript, the interesting science, and the quality of the writing. The three referees raised a number of scientific points, which the authors have addressed thoroughly in each individual case. Indeed their professional response is to be commended. I believe the manuscript is now ready for publication and I recommend acceptance in its current form.

Reviewer #2 (Remarks to the Author):

I thank the Authors for their very detailed response and I fully support the publication of their manuscript

Reviewer #3 (Remarks to the Author):

I think that the authors have fully answered the questions posed in my first reports, and, as far as I can tell, they have also answered criticisms from the other referees.
I then advise to publish the paper.