nature portfolio

Peer Review File

Activating non-Hermitian skin modes by parity-time symmetry breaking

Reviewers' comments:

Reviewer #1 (Remarks to the Author):

In this work, the authors investigate different types of the non-Hermitian skin effect (NHSE) associated with PT-symmetry breaking in several non-Hermitian lattice models. The switching of various skin modes is studied by selectively activating the skin effect in the presence of PT symmetry. These skin modes, with different origins, include corner modes and hybrid skin-topological modes in the presence of topological edge states.

I find that the paper is overall very clearly written and accessible to a general audience. Considering that NHSE and PT symmetry breaking are important issues in current research, I think the results presented in this work should be of broad interest. However, I am not fully convinced that the results reported in this work represent a significant advancement in the field of non-Hermitian physics. For example, the key ingredient in the phenomena studied in this work, namely PT-symmetry breaking, has been thoroughly explored in the literature. The notion of hybrid skin-topological modes is barely new at this point. Therefore, before suggesting its publication, I need a stronger argument for the significant conceptual or technical progress in this work to warrant publication in Communication Physics.

A technical issue on page 4, line 5, concerns how to obtain the edge Hamiltonian via the projection operator. Usually, in Hermitian systems, the edge Hamiltonian is obtained using a Wilson-loop technique, as in L. Fidkowski et al., Phys. Rev. Lett. 107, 036601 (2011). This method involves the calculation of Berry connections. In non-Hermitian systems, it is not clear how the Berry connection is constructed (for both left and right eigenstates?) and how the Wilson loop relates to the edge Hamiltonian.

Reviewer #2 (Remarks to the Author):

The authors propose the way to control the localization by the non-Hermitian skin effect via the modulation of symmetries. It is useful to utilize the PT symmetry because the symmetry breaking can switch the spectra of non-Hermitian systems from real to complex, which leads to the non-Hermitian skin effect. Based on this mechanism, the authors demonstrate the various types of localization behaviors in two and three dimensional non-Hermitian systems.

I think that the demonstration of manipulating the non-Hermitian skin effect is interesting because the localization phenomena are unique to higher-dimensional non-Hermitian systems, which have attracted much attention in recent researches of non-Hermitian physics. However, I still wonder whether it is appropriate to call the phenomena to "PT-activated" or not.

Indeed, the authors confuse with the two types of PT symmetries. One is the conventional PT symmetry, which is strictly defined by symmetries that a Hamiltonian and its eigenstates. Another one is the non-Bloch PT symmetry. Here, I do not understand the clear definition of the non-Bloch PT symmetry. The authors said that, when the whole spectra of non-Bloch Hamiltonians become real, the systems described by the Hamiltonians protect the non-Bloch PT symmetry. Wouldn't this condition be a necessary condition instead of a sufficient condition? I think that the demonstration proposed here does not originate from "PT-symmetry" breaking until the clear definition of the non-Bloch PT symmetry is given. Indeed, without such definition, one cannot how to construct non-Hermitian systems with the "PT symmetry".

From the above viewpoint, I would not like to publish this manuscript for Communications Physics.

Reviewer #3 (Remarks to the Author):

REPORT

The manuscript describes the possibility of generating various classes of non-Hermitian skin effects (NHSEs) in non-Hermitian (NH) systems via breaking their parity-time (PT) symmetry locally, (i.e., in some of the systems sectors, like edges), instead of breaking it globally, which is usually assumed in such studies.

Let me stress that the topic of NHSEs has been attracting a rapidly growing interest, as can be seen by reading the very recent reviews [24,25] and papers cited there, including a related paper on "Topological Origin of Non-Hermitian Skin Effects" by Okuma et al. published in 2020, which has already 650+ citations according to Google Scholar.

Although I am not fully convinced about the validity of studying quantum NH systems when the effect of quantum jumps is totally ignored (see my remarks below), but I must admit that such a quantum-jump-free approach is common in this field.

The reported results in the manuscript seem to be novel and I believe that they can stimulate some further related research studies. Thus, I could recommend the publication of the manuscript in Communications Physics if the Authors adequately revised their paper to address the following comments:

1. The applied models include losses in a semiclassical way via non-Hermitian Hamiltonians *without* quantum jumps. To properly describe open quantum systems, quantum jumps, representing the instantaneous switching between the energy levels of the system, should be included. This could be done by analyzing a quantum Liouvillian (corresponding to a master equation) as described in:

[E1] F. Minganti et al., Quantum exceptional points of non-Hermitian Hamiltonians and Liouvillians: The effects of quantum jumps, Phys. Rev. A 100, 062131 (2019).

We note that the NH-SSH Hamiltonian, as given in (1), can be identified as a part of the corresponding master equation (and, thus, a Liouvillian), in addition to extra quantum-jump terms.

I do not insist on performing new calculations for the present manuscript, but I think it would be beneficial for the Readers of the paper, to read two or three sentences in the Introduction about this issue with the reference to [E1].

2. Please cite (in addition to [36]) the original paper of Su, Schrieffer, and Heeger, where the famous model was introduced, i.e.:

[E2] W. P. Su, J. R. Schrieffer, and A. J. Heeger Solitons in Polyacetylene, Phys. Rev. Lett.. 42, 1698 (1979).

3. It is known, see, e.g.,

[E3] Ch.-Y. Ju et al., Flattening the Curve with Einstein's Quantum Elevator: Hermitization of Non-Hermitian Hamiltonians via the Vielbein Formalism, Phys. Rev. Research 4, 023070 (2022).

that "any non-Hermitian Hamiltonian can be transformed into a Hermitian one without altering the physics." The transformation is unitary and is referred to as a "vielbein" transformation.

My question is if one unitarily transforms the studied NH SSH Hamiltonian, as given in Eq. (1), into a Hermitian Hamiltonian, can we still observe the described skin effects?

Again, I do not insist on performing any new calculations for the present manuscript, as it might be quite demanding to find the required vielbein. Anyway, I think it would be useful for the readers to read a few sentences in, e.g., the Introduction on the existence of such transformations for the Hermitization of the models studied in the manuscript.

4. Please add a reference to the applied definition of the fractal dimension in Eq. (2).

5. The notation |\psi| in Eq. (2) might suggest that the absolute-value norm is applied. A reader could expect that the standard vector norm ||\psi|| should be used, i.e., the square root of the inner product of the vector and itself. Anyway, please specify the applied norm.

6. Four equations in Eq. (14) can be much simplified by introducing the auxiliary functions $W_{\pm}=\sqrt{(u\pm\gamma/2)/(u\mp\gamma/2)}$.

7. Four equations in Eq. (16) can be simplified by introducing the auxiliary function $D=1/sqrt{C^2-B^2}$.

8. Eqs. (18) and (20) can be simplified by using the auxiliary symbol $E\equiv A^2+C^2-B^2-D^2\ge 0$, we can be introduced below Eq. (18).

Of course, the suggested symbols W,D,E can be replaced by other letters.

9. Exceptional points should be mentioned, especially in their relation to the transition between the PT and broken-PT regimes. I have noted that a number of references on exceptional points have already been cited although, the points are not mentioned explicitly in the main text.

Minor comments:

10. The phrase "NHSE phenomena" is often used, but it literally means "non-Hermitian skin effect phenomena". Thus, I would suggest to abbreviate the phrase to "non-Hermitian skin effects" or

"NHSEs". Note that the term "non-Hermitian skin effects" is commonly used, e.g., in [29].

11. The applied notation:

(1) "γ (g) [is] the asymmetric hopping (gain/loss) parameter"

might be confusing for many readers, which usually denote the quantities in the opposite way, i.e.:

(2) "g (γ) is the asymmetric hopping (gain/loss) parameter$".

I understand that it is now too late to swap the notations (1) and (2), but maybe the Authors can consider at least to denote the gain/loss parameter in a more standard way (e.g., by λ, if not κ, which is already used to denote another quantity in the paper).

12. Please add mathematical definitions of the transformation \mathcal{PT} or, equivalently, \mathcal{K}.

13. Fonts used in figures (especially in Fig. 4) are too small.

14. Define the symbols CP in the abstract and page 3. Note that now it is defined only on page 5.

15. To avoid confusion, instead of referring to "Supplementary Information 2", "Supplementary Information 1A", etc., better to write "Sec. 2 in the Supplementary Material", etc.

Author Responses to All Reviewers (**COMMSPHYS-23-0890-T**)

---Author Responses to Reviewer # 1 (after citing referee's comments first): ---

In this work, the authors investigate different types of the non-Hermitian skin effect (NHSE) associated with PT-symmetry breaking in several non-Hermitian lattice models. The switching of various skin modes is studied by selectively activating the skin effect in the presence of PT symmetry. These skin modes, with different origins, include corner modes and hybrid skin-topological modes in the presence of topological edge states.

However, I am not fully convinced that the results reported in this work represent a significant advancement in the field of non-Hermitian physics. For example, the key ingredient in the phenomena studied in this work, namely PT-symmetry breaking, has been thoroughly explored in the literature. The notion of hybrid skin-topological modes is barely new at this point. Therefore, before suggesting its publication, I need a stronger argument for the significant conceptual or technical progress in this work to warrant publication in Communication Physics.

We thank our reviewer for his/her precious time in reviewing out work. Indeed, since phenomena such as PT-symmetry breaking and hybrid skin-topological effect (HSTE) are already known in the literature, the novelty and significant progress in this work may not be obvious in the previous version of our manuscript. To address this concern, we would like to clarify and emphasize that the significant progress of our work mainly lies in the following two directions:

1). We have developed a unified paradigm to design higher-dimensional systems with different types of bulk/surface/edge non-Hermitian skin effect (NHSE) as desired. This represents a significant advance because the <u>combination of various types of NHSE is highly nontrivial due to the non-locality of the NHSE, particularly in higher dimensions</u>. As a case in point, when extended to higher dimensions, even relatively simple models can acquire enigmatic new types of NHSE effects from the higher-dimensional interplay, for instance in [1-2].

In other words, many of the results in our work cannot be predicted just based on the understanding of the individual mechanisms. As an analogy, even though the usual 1D NHSE and 1D SSH topological localization are well-known, the HSTE that arise from their 2D interplay is a nontrivial consequence.

The PT-symmetry breaking (or the breakdown of other pseudo-Hermiticity), which is a cornerstone on which our paradigm are built, showcases how PT symmetry breaking can lead to various different new states beyond just controlling the reality of the spectrum. The thorough exploration of it also makes our results more accessible to the readers.

2). With this new paradigm, we have also discovered several new phenomena that are previously unknown, including the directional toggling of NHSE, and the non-monotonic dynamics of a "reversal" of HSTE, where NHSE is absent in one branch of edge states but presents in the bulk. The HSTE, which "is barely new at this point", is only one of the examples to show the versatility of our paradigm, and even in itself possesses some previously unknown properties: the HSTE designed with our scheme shows a geometry-independent [1] feature of its skin-free bulk states, which is absent in those studied in existing literature [3-6]. We emphasis here, the occurrence of bulk NHSE is a rather universal phenomenon for two- or higher-dimensional non-Hermitian Hamiltonians, as their complex bulk spectra generally cover certain area in the complex plane [1]. In previous works on HSTE [3-6], non-reciprocities (which induces NHSE) cancel each other only along certain directions (e.g, x- and y-direction), thus bulk states are NHSE-free only under certain geometries. It is a nontrivial task to eliminate net non-reciprocity along arbitrary directions in the bulk but not in the boundaries (so to have HSTE), and our work shows it can be realized in PT symmetric systems.

We regret that the significance of our work has not been apparent to the reviewer previously. In the our revision, we have revised the introduction and discussion sections to highlight the significance of our paradigm, that it not only leads to the discovery of several new phenomena of NHSE, but also provides a useful scheme for constructing models with different types of NHSE. In fact, the significance of our work is also acknowledged by our Reviewer #3, who thinks our results are "novel" and "can stimulate some further related research studies". Reviewer #2 also thinks our work "interesting". We hope with our revisions, the originality and usefulness of our work will become much clearer, and make a clear case for publication in Communications Physics.

A technical issue on page 4, line 5, concerns how to obtain the edge Hamiltonian via the projection operator. Usually, in Hermitian systems, the edge Hamiltonian is obtained using a Wilson-loop technique, as in L. Fidkowski et al., Phys. Rev. Lett. 107, 036601 (2011). This method involves the calculation of Berry connections. In non-Hermitian systems, it is not clear how the Berry connection is constructed (for both left and right eigenstates?) and how the Wilson loop relates to the edge Hamiltonian.

We thank Reviewer for pointing out this reference. This reference relates the spectrum of boundary states to the Berry connection of bulk states in Hermitian systems. This method can also be used for our non-Hermitian systems. Actually, 2D Hamiltonian H_{2D,g} in Eq (5) can be considered as a 1D system along x-direction with external parameter k_y. Under this interpretation, the appearance of boundary states along x-boundaries can be anticipated by the Wilson loop spectrum (Zak phase) calculated with biorthogonal eigenvectors [7]. For the situation shown in Figs.3(a) and 3(b), we find that the Zak phase is \pi for all values of k_y, indicating in-gap boundary states appear for any value of k_y. Furthermore, the energies of boundary states fall in the center of bulk spectrum, corresponding to the quantization of Zak phase at \pi.

After ensuring the appearance of boundary states through Wilson loop, we use a projector to obtain their effective edge Hamiltonian to analyze the edge spectrum. This projector is built on the polarization properties of boundary states. For instance, each boundary state of the Hermitian Su-Schrieffer-Heeger model occupies only one sublattice, and the projectors are thus given by

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ for the two boundary states respectively. The effective edge Hamiltonian is then obtained by acting this projector on the original Hamiltonian. This method can be extend into multiband system and agrees with the discussion of [7], and has also been used to construct higher-order topological phases [9-11].

As a clarification, we have added these references in our revision and modified the discussion of effective Hamiltonian of boundary states accordingly in the second paragraph of page 4 (after Eq. 5).

References:

[1] K. Zhang, Z. Yang, and C. Fang, Nat. Commun. 13, 2496 (2022).

[2] H. Jiang and C. H. Lee, Phys. Rev. Lett. 131, 076401 (2023).

[3] C. H. Lee, L. Li, and J. Gong, Phys. Rev. Lett. 123, 016805 (2019).

[4] L. Li, C. H. Lee, and J. Gong, Phys. Rev. Lett. 124, 250402 (2020).

[5] Y. Li, C. Liang, C. Wang, C. Lu, and Y.-C. Liu, Phys. Rev. Lett. 128, 223903 (2022).

[6] W. Zhu and J. Gong, Phys. Rev. B 106, 035425 (2022)

[7] D. C. Brody, J. Phys. A: Math. Theor. 47, 035305 (2013).

[8] R. S. K. Mong and V. Shivamoggi, Phys. Rev. B 83, 125109 (2011).

[9] M. Geier, L. Trifunovic, M. Hoskam, and P. W. Brouwer, Phys. Rev. B 97, 205135 (2018).

[10] L. Trifunovic and P.W. Brouwer, Phys. Rev. X 9, 011012 (2019).

[11] Z. Lei, Y. Deng, and L. Li, Phys. Rev. B 106, 245105 (2022).

---Author Responses to Reviewer # 2 (after citing referee's comments first): ---

The authors propose the way to control the localization by the non-Hermitian skin effect via the modulation of symmetries. It is useful to utilize the PT symmetry because the symmetry breaking can switch the spectra of non-Hermitian systems from real to complex, which leads to the non-Hermitian skin effect. Based on this mechanism, the authors demonstrate the various types of localization behaviors in two and three dimensional non-Hermitian systems.

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Indeed, the authors confuse with the two types of PT symmetries. One is the conventional PT symmetry, which is strictly defined by symmetries that a Hamiltonian and its eigenstates. Another one is the non-Bloch PT symmetry. Here, I do not understand the clear definition of the non-Bloch PT symmetry. The authors said that, when the whole spectra of non-Bloch Hamiltonians become real, the systems described by the Hamiltonians protect the non-Bloch PT symmetry. Wouldn't this condition be a necessary condition instead of a sufficient condition? I think that the demonstration proposed here does not originate from "PT-symmetry" breaking until the clear definition of the non-Bloch PT symmetry is given. Indeed, without such definition, one cannot how to construct non-Hermitian systems with the "PT symmetry".

From the above viewpoint, I would not like to publish this manuscript for Communications Physics.

We thank the reviewer for reviewing our work and acknowledging that our results are "interesting". Indeed, our Reviewer #3 also thinks our results are "novel" and "can stimulate some further related research studies. We are also grateful to the reviewer for providing this very important comment. In our previous manuscript, the explicit definition non-Bloch PT symmetry is discussed in the Methods section, but we now realize that this is not so friendly to the readers, as this symmetry is less well-known and may be confused with the conventional PT symmetry. Also, it was admittedly a bit confusing since its explicit definition involves an auxiliary Hamiltonian (the non-Bloch Hamiltonian) and a momentum-dependent symmetry operator, which make it less feasible for manipulating the symmetry and constructing corresponding symmetric models.

Having this in mind, we reexamine our models and their symmetries, and find that the they also satisfy a generalized PT symmetry [1,2], which take a much simpler form compared to the non-Bloch version. Hence we have thoroughly revised our presentation and definition of PT symmetry used in this work, and explicitly expressed it in terms of a generalized PT symmetry. Since this symmetry also can protect OBC real spectra, we now give its clear definition in the main text, and refer to it as the symmetry associated with the NHSE for the model $H_{\gamma,2D}$ of Eq. 1 in the manuscript. The discussion of non-Bloch PT symmetry is still kept in the Methods section, as it is a more universal symmetry that protects real OBC spectrum for generic non-Hermitian systems.

Below, for completeness, we clarify the three types of "PT" symmetries mentioned above. The conventional Bloch PT symmetry is defined as a combination of space-inverse and time-inverse symmetry, leading to

$$U_{PT}\mathcal{K}H(k)[U_{PT}\mathcal{K}]^{-1} = H(k),$$

where $\mathcal{K}$ is the complex conjugation operator and U_{PT} is an unitary operator. It is well known that H(k) can hold real spectrum when its parameters fall in PT-symmetry unbroken regime, even for non-Hermitian H(k).

On the other hand, as discussed in the revised manuscript, by taking ky as a parameter, the Hamiltonian $H_{\gamma,2D}$ in Eq. 1 of the manuscript only have real-valued hopping parameters. Thereby, $H_{\gamma,2D}$ in real space also holds a (generalized) PT symmetry, read as [1,2]:

$$\mathcal{K}H_{\gamma,2D}\mathcal{K} = H_{\gamma} ,$$

which also can protect real spectrum as it satisfies the form of PT symmetry in real space, where the momentum k is absent in the Hamiltonian [1,2]. Note that this expression is also equivalent to the spinless time-reversal symmetry in real space.

One major difference of these two types of PT symmetries is the second one not only can be broken by tuning parameters, but also by changing the boundary conditions. As seen in Fig.5(c) of our manuscript, the spectrum under open boundary condition (periodic boundary condition) is real (complex). In this sense, this type of PT symmetry is induced from non-Hermitian skin effects (NHSE) [2-5], and has been experimentally observed [6]. Therefore, it can be defined in generalized Brillouin zone (GBZ) associating with non-Bloch states, which provides an effective "Bloch-like" Hamiltonian to describe the OBC spectrum of the system, as discussed in our Methods section. After introducing the transition: $k_x \to k_x + i\kappa$, the GBZ Hamiltonian satisfies

$$\eta_{PT}\mathcal{K}H_\gamma(k_x + i\kappa)[\eta_{PT}\mathcal{K}]^{-1} = H_\gamma(k_x + i\kappa),$$

Which takes the same form as the conventional PT symmetry, indicating the real spectrum can appear under open boundary condition.

To sum up, our model satisfies both the generalized PT symmetry and the non-Bloch PT symmetry. The first one is more suitable for constructing non-Hermitian systems with the PT symmetry (as it only requires the matrix elements to be real). In contrast, while the non-Bloch PT symmetry is more generic (without the restriction of real-valued elements), it explicit form is much more complicated, making it difficult for manipulating the symmetry of the system. To this regard, <u>we have now mainly discussed the generalized PT symmetry (with its clear definition) in our main text</u>, and provided the explicit form of the non-Bloch PT symmetry in the Methods section.

References:

[1] C. M. Bender, M. V. Berry, and A. Mandilara, Journal of Physics A: Mathematical and General 35, L467-L471 (2002).

[2] F. Song, H.-Y. Wang, and Z. Wang, in A Festschrift in Honor of the C. N. Yang Centenary, (2022), pp. 299–311.

[3] S. Yao, and Z. Wang, Phys. Rev. Lett. 121, 086803 (2018).

[4] S. Longhi, Optics Letters 44, 5804 (2019).

[5] S. Longhi, Phys. Rev. Research 1, 023013 (2019).

[6] L. Xiao, T. Deng, K. Wang, Z. Wang, W. Yi, and P. Xue, Phys. Rev. Lett. 126, 230402 (2021).

---Author Responses to Reviewer # 3 (after citing referee's comments first): ---

The manuscript describes the possibility of generating various classes of non-Hermitian skin effects (NHSEs) in non-Hermitian (NH) systems via breaking their parity-time (PT) symmetry locally, i.e., in some of the systems sectors, like edges), instead of breaking it globally, which is usually assumed in such studies.

Let me stress that the topic of NHSEs has been attracting a rapidly growing interest, as can be seen by reading the very recent reviews [24,25] and papers cited there, including a related paper on "Topological Origin of Non Hermitian Skin Effects" by Okuma et al. published in 2020, which has already 650+ citations according to Google Scholar.

Although I am not fully convinced about the validity of studying quantum NH systems when the effect of quantum jumps is totally ignored (see my remarks below), but I must admit that such a quantum-jump-free approach is common in this field.

The reported results in the manuscript seem to be novel and I believe that they can stimulate some further related research studies. Thus, I could recommend the publication of the manuscript in Communications Physics if the Authors adequately revised their paper to address the following comments:

We are grateful the Reviewer for the careful reading and the recommendation of our work. We are also glad to learn that the Reviewer shares our view of the scientific significance of this work, that our results are "novel" and "can stimulate some further related research studies". Below we will carefully address all the comments.

1.The applied models include losses in a semiclassical way via non-Hermitian Hamiltonians *without* quantum jumps. To properly describe open quantum systems, quantum jumps, representing the instantaneous switching between the energy levels of the system, should be included. This could be done by analyzing a quantum Liouvillian (corresponding to a master equation) as described in:

[E1] F. Minganti et al., Quantum exceptional points of non-Hermitian Hamiltonians and Liouvillians: The effects of quantum jumps, Phys. Rev. A 100, 062131 (2019).

We note that the NH-SSH Hamiltonian, as given in (1), can be identified as a part of the corresponding master equation (and, thus, a Liouvillian), in addition to extra quantum-jump terms.

I do not insist on performing new calculations for the present manuscript, but I think it would be beneficial for the Readers of the paper, to read two or three sentences in the Introduction about this issue with the reference to [E1].

We thank the Reviewer for pointing out this reference. We have added two sentences in the third paragraph to discuss the scope of effective non-Hermitian Hamiltonian, with proper references to [E1] and some related papers. We also note here, the models discussed in this work can also be implemented with electrical circuits [1-5] where various types of non-Hermitian skin effects have been observed.

2. Please cite (in addition to [36]) the original paper of Su, Schrieffer, and Heeger, where the famous model was introduced, i.e.:

[E2] W. P. Su, J. R. Schrieffer, and A. J. Heeger Solitons in Polyacetylene, Phys. Rev. Lett. 42, 1698 (1979).

We have cited this paper in the sentence describing SSH model in this revised manuscript.

3. It is known, see, e.g., [E3] Ch.-Y. Ju et al., Flattening the Curve with Einstein's Quantum Elevator: Hermitization of Non-Hermitian Hamiltonians via the Vielbein Formalism, Phys. Rev. Research 4, 023070 (2022). that "any non-Hermitian Hamiltonian can be transformed into a Hermitian one without altering the physics." The transformation is unitary and is referred to as a "vielbein" transformation.

My question is if one unitarily transforms the studied NH SSH Hamiltonian, as given in Eq. (1), into a Hermitian Hamiltonian, can we still observe the described skin effects?

Again, I do not insist on performing any new calculations for the present manuscript, as it might be quite demanding to find the required vielbein. Anyway, I think it would be useful for the readers to read a few sentences in, e.g., the Introduction on the existence of such transformations for the Hermitization of the models studied in the manuscript.

We thank the Reviewer for pointing out this reference. In this reference, the authors revealed that non-Hermitian Hamiltonian can be transformed into Hermitian Hamiltonian with operator called as “vielbein”. For the non-Hermitian Hamiltonian with non-Hermitian skin effect (NHSE), this transformation also exists. In a previous work [6], it has been shown that the non-Hermitian SSH model with asymmetric hoppings holding NHSE can be transformed into Hermitian SSH model without NHSE through a similarity transformation. Instead, the NHSE of the original non-Hermitian model is embodied in the transformation operator, whose element is space dependent. More generally, non-Hermitian Hamiltonians with NHSE describing by the generalized Brillouin zone [7-9] can be mapped to Hamiltonians without NHSE through a complex deformation of the crystal momentum, which is equivalent to a similarity transformation. Therefore, the non-Hermitian Hamiltonian with NHSE can be transformed to a Hermitian Hamiltonian, considering Hermitization of non-Hermitian Hamiltonians without NHSE have been constructed in [Phys. Rev. Research 4, 023070 (2022)].

We added this paper to the reference and a sentence in the first paragraph of Method in this revised manuscript..

4. Please add a reference to the applied definition of the fractal dimension in Eq. (2).

We have cited [10] in the sentence before giving Eq.(2) of this revised manuscript.

5. The notation |\psi| in Eq. (2) might suggest that the absolute-value norm is applied. A reader could expect that the standard vector norm ||\psi|| should be used, i.e., the square root of the inner product of the vector and itself. Anyway, please specify the applied norm.

In Eqs.(2) and (4), \psi_{n,\mathbf{r}} actually represent the amplitude of n-th eigenstate at r-site. The notation |\psi_{n,\mathbf{r}}| is its absolute value.

In this revised manuscript, we have added sentence after Eq.(2) to explain this definition.

6. Four equations in Eq. (14) can be much simplified by introducing the auxiliary functions $W_{\pm}=\sqrt{(u\pm\gamma/2)/(u\mp\gamma/2)}$.

7. Four equations in Eq. (16) can be simplified by introducing the auxiliary function $D=1/sqrt{C^2-B^2}$.

8. Eqs. (18) and (20) can be simplified by using the auxiliary symbol $E\equiv A^2+C^2-B^2-D^2\ge 0$, we can be introduced below Eq. (18). Of course, the suggested symbols W,D,E can be replaced by other letters.

We thank the Reviewer for these helpful suggestions. We have introduced $W_{\pm}=\sqrt{(u\pm\gamma/2)/(u\mp\gamma/2)}$, $F=1/\sqrt{C^2-B^2}$ and $G=A^2+C^2-B^2-D^2$ in expressing these equations.

9. Exceptional points should be mentioned, especially in their relation to the transition between the PT and broken-PT regimes. I have noted that a number of references on exceptional points have already been cited although, the points are not mentioned explicitly in the main text.

We have added a sentence in the last paragraph of page 2 to describe the exceptional points, where the spectra under PT unbroken and broken regime are compared.

Minor comments:

10. The phrase "NHSE phenomena" is often used, but it literally means "non-Hermitian skin effect phenomena". Thus, I would suggest to abbreviate the phrase to "non-Hermitian skin effects" or "NHSEs". Note that the term "non-Hermitian skin effects" is commonly used, e.g., in [29].

We have replaced the phrase "NHSE phenomena" with more appropriate ones in our manuscript.

11. The applied notation:

(1) "γ (g) [is] the asymmetric hopping (gain/loss) parameter" might be confusing for many readers, which usually denote the quantities in the opposite way, i.e.:

(2) "g (γ) is the asymmetric hopping (gain/loss) parameter$". I understand that it is now too late to swap the notations (1) and (2), but maybe the Authors can consider at least to denote the gain/loss parameter in a more standard way (e.g., by λ, if not κ, which is already used to denote another quantity in the paper).

We thank the reviewer for pointing out this subtle issue. It is indeed confusing that different symbols may be used to denote the same quantity (or vice versa) in literature. In the context of gain and loss, we note that γ has also been used for describing asymmetric hoppings, including the first paper on NHSE [6]. As a matter of fact, asymmetric hoppings can also been interpreted as non-local gain and loss, and can be mapped to local (on-site) gain and loss for the SSH model through a basis change $\sigma_z\leftrightarrow\sigma_y$, as discussed in Ref. [11]. On the other hand, we note there g has also been used to denote on-site gain and loss in literature, for examples, Refs. [12,13]. For these reasons, we would like to keep the current notations of parameters in our manuscript. On the other hand, in the Methods section, we have revised the explanation of these notations to point out that both g and

γ can be interpreted as certain types of dissipation (referring to Ref. [9]). We hope this revision can lift the reviewer's concern.

12. Please add mathematical definitions of the transformation \mathcal{PT} or, equivalently, \mathcal{K}.

We have added explanation of the complex conjugate operator $\mathcal{K}$ in the second paragraph of page 4.

13. Fonts used in figures (especially in Fig. 4) are too small.

We have now enlarged some texts in Figs. 1 and 4. In addition, Fig. 4 is now made a two-column figure, so that the texts within can be clearly seen.

14. Define the symbols CP in the abstract and page 3. Note that now it is defined only on page 5.

We have added the definition: charge-parity ($\mathcal{CP}$) symmetry, in abstract and introduction instead of page 5.

15. To avoid confusion, instead of referring to "Supplementary Information 2", "Supplementary Information 1A", etc., better to write "Sec. 2 in the Supplementary Material", etc.

We have taken our Reviewer's advice, and changed associating sentences.

References:

[1] Tobias Helbig, Tobias Hofmann, S Imhof, M Abdelghany, T Kiessling, LWMolenkamp, C. H. Lee, A Szameit, M Greiter, and R Thomale, Nature

Physics 16, 747-750 (2020)

[2] D. Zou, T. Chen, W. He, J. Bao, C. H. Lee, H. Sun, and X. Zhang, Nature Communication 12, 7201 (2021).

[3] Ce Shang, et al., Advanced Science 9, 2202922 (2022).

[4] H. Zhang, T. Chen, L. Li, C. H. Lee, and X. Zhang, Phys. Rev. B 107, 085426 (2023).

[5] Z., Xiao, et al., arXiv: 2211.09152 (2022).

[6] S. Yao, and Z. Wang, Phys. Rev. Lett. 121, 086803 (2018).

[7] K. Yokomizo and S. Murakami, Phys. Rev. Lett. 123, 066404 (2019).

[8] C. H. Lee and R. Thomale, Phys. Rev. B 99, 201103(R) (2019).

[9] R. Lin, T. Tai, L. Li, and C. H. Lee, Frontiers of Physics 18, 53605 (2023).

[10] J. Theiler, J. Opt. Soc. Am. A 7, 1055 (1990).

[11] F. Song, S. Yao, and Z. Wang, Phys. Rev. Lett. 123, 170401 (2019).

[12] S. Liu, S. Ma, C. Yang, et al., Phys. Rew. Applied 13, 014047 (2020).

[13] A. Melkani, Phys. Rev. Research, 5, 023035 (2023).

Reviewers' comments:

Reviewer #1 (Remarks to the Author):

In their response, the authors addressed my earlier concerns regarding the "novelty" and "significant advancement" of the results, persuading me of their work's scientific merit and appeal within the non-Hermitian community. I agree with the assertion that "many of the results in our work cannot be predicted just based on the understanding of the individual mechanisms" and appreciate their revisions in the introduction and discussion sections to emphasize their paradigm. Hence, I recommend its publication in Commun. Phys. in its current form. The explanations in response to Review #2 regarding PT-symmetry are satisfactory to me. The authors have also appropriately addressed the comments from Review #3. Concerning the "vielbein" transformation, I perceive it as a non-local transformation in real space designed to nullify the spectral pumping effect? Besides, non-Hermitian physics, it is usually treated as an independent, albeit interrelated, topic with open system described by master equation.

Reviewer #2 (Remarks to the Author):

I would like to thank the authors for examining my concerns.

I'm afraid I have to disagree with the authors' opinions. First, there are no clear reasons to refer to a time-reversal symmetry as a "generalized" PT symmetry. Therefore, it is still unclear whether the phenomena proposed here can be induced by PT symmetries. In my opinion, it seems that the non-Hermitian skin effect merely originates from the change in the boundary conditions. Second, the eigenvalues under full open boundary conditions do not always take real values, even if hopping amplitudes of non-Hermitian Hamiltonians are real. This indicates that the proposed theory would not function to design desired non-Hermitian models.

I feel that the authors do not appropriately answer my questions about the definitions of PT symmetries. Namely, what are necessary and sufficient conditions for the non-Bloch PT symmetry? I think that the paper should not be published in Communications Physics unless the authors clarify the above questions.

Reviewer #3 (Remarks to the Author):

The Authors have adequately revised their manuscript to address the comments of my first report. I find the revised version considerably improved. In particular, the Authors have more explicitly related their results to those reported in the literature including, e.g., the role of quantum jumps, exceptional points, or the Hermitization of the studied non-Hermitian system and related ones. Thus, I can recommend the publication of the revised manuscript, in its present form, in Communications Physics.

Author Responses to All Reviewers **(COMMSPHYS-23-0890A)**

We thank all the reviewers for their efforts in reviewing our manuscript. In the second round of review, reviewers #1 and #3 have recommended publication of our work in in Commun. Phys., while reviewer #2 requires for further explanation about the non-Bloch PT symmetry and some related issues. In addition, reviewer #1 have also commented on the vielbein transformation mentioned by reviewer #3:

"*Concerning the "vielbein" transformation, I perceive it as a non-local transformation in real space designed to nullify the spectral pumping effect? Besides, non-Hermitian physics, it is usually treated as an independent, albeit interrelated, topic with open system described by master equation.*"

We agree with the reviewer's comment. However, since it is not the focus of our paper, we decide not to add further explanation regarding the vielbein transformation and the difference between non-Hermitian physics and open systems.

Below we provide our point-by-point response to reviewer #2.

---Author Responses to Reviewer # 2 (after citing referee's comments first): ---

I'm afraid I have to disagree with the authors' opinions. First, there are no clear reasons to refer to a time-reversal symmetry as a "generalized" PT symmetry. Therefore, it is still unclear whether the phenomena proposed here can be induced by PT symmetries.

We thank the Reviewer for this sharp comment. We first would like to point out that both the "generalized" PT and non-Bloch PT symmetries are extensions of the "conventional" PT symmetry, as they can be mapped to the latter with certain transformation of the Hamiltonian matrix and protect real eigenenergies in their symmetry-unbroken phases, as explained below.

1). The non-Bloch PT symmetry of our model is explicitly discussed in the method section. To summarize, this symmetry is defined for the non-Bloch Hamiltonian $H(k+i\kappa)$ that describes the OBC spectrum of the non-Hermitian Hamiltonian. In our non-Hermitian SSH model, the non-Bloch symmetry takes the form of $[\sigma_x'\mathcal{K}, H(k+i\kappa)]=0$ with σ_x' a two-by-two matrix satisfying $(\sigma_x'\mathcal{K})^2=1$, analogous to the convectional PT symmetry of the Hermitian SSH model, $[\sigma_x\mathcal{K}, H_{SSH}(k)]=0$.

2). The terminology "generalized PT symmetry" is firstly used in Ref [1], which discusses the relation between real spectra and anti-unitary symmetry $A = U\mathcal{K}$ with U an unitary operator and $\mathcal{K}$ the complex conjugation operator. The conclusion is that real spectra are protected in the symmetry unbroken regime of the Hamiltonian, as long as the anti-unitary symmetry satisfied $A^{2m} = 1$ with m being odd integer.

In our case, after k_y as a parameter, the Hamiltononian $H_{\gamma,2D}$ in Eq. 1 of the manuscript takes real-valued hopping parameters, and satisfies the above definition with A being the identity matrix. The corresponding generalized PT symmetry is the same as the one discussed in Ref. [2] for a non-Hermitian SSH model. The mapping between this generalized PT symmetry and the conventional one can be seen by changing the basis as $|u_r, A/B\rangle = \frac{1}{2}\left[(1-i)|-r, A/B\rangle + (1+i)|r, A/B\rangle\right]$, with $|r, A/B\rangle$ denoting the real space basis with A/B the sublattices and r the position of unit cells along x direction. In the new basis, the Hamiltonian matrix satisfies $\langle u_r, \alpha|H_{\gamma,2D}|u_l, \beta\rangle = \left(\langle u_{-r}, \alpha|H_{\gamma,2D}|u_{-l}, \beta\rangle\right)^*$, i.e. it is symmetric under the combination of spatial-inversion (r -> -r) and spinless time-reversal (complex conjugation) operations in the new basis.

Due to the similarity between the conventional PT symmetry and these two symmetries, we can conclude that the system shall have real spectrum in the unbroken regimes of these two symmetries. In particular, the generalized PT symmetry in our model is determined by the real-valued parameters (in the momentum space of k_y), thus the Hamiltonian always satisfies this symmetry, and can have complex eigenenergies only in the symmetry broken regime. On the other hand, with further investigation into the non-Bloch PT symmetry, we find that complex eigenenergies emerge when this symmetry becomes invalid for our model, which is now discussed more explicitly in the Method section. To avoid confusion, we have also removed the statement "equivalent to the spinless time-reversal symmetry" when introducing the generalized PT symmetry in the main text.

In my opinion, it seems that the non-Hermitian skin effect merely originates from the change in the boundary conditions.

Throughout our manuscript, we have displayed various examples showing how different NHSE correspond to the symmetry conditions. The simplest example for answering this question is the one shown in Fig. 1 in the manuscript. That is, the generalized/non-Bloch PT symmetry is broken/invalid when $k_y \in (0, \pi)$, and unbroken/valid when $k_y \in [-\pi, 0]$. The NHSE corresponding to the states in these two regions also show different behaviors of corner- and edge-localization, while the system is under the *same* open boundary conditions.

We have revised related discussion for a better clarification this details.

Second, the eigenvalues under full open boundary conditions do not always take real values, even if hopping amplitudes of non-Hermitian Hamiltonians are real. This indicates that the proposed theory would not function to design desired non-Hermitian models.

We would like to emphasize that the key idea of our proposed theory is that PT symmetry (or other pseudo-Hermiticity such as generalized and non-Bloch PT symmetries) ensures real spectrum in its unbroken phase, thus eliminates NHSE. That is to say, we can tune the system's parameters to the PT unbroken/broken phases (the asymmetric phase for non-Bloch PT symmetry), and deactivate/activate the NHSE. The reviewer's observation actually verifies our theory: the eigenvalues under full OBCs may take complex values even if hopping amplitudes are real, which means that the system is in the (generalized) PT-broken regime and supports NHSE different from that in the unbroken regime.

I feel that the authors do not appropriately answer my questions about the definitions of PT symmetries. Namely, what are necessary and sufficient conditions for the non-Bloch PT symmetry? I think that the paper should not be published in Communications Physics unless the authors clarify the above questions.

We apologize for haven't provided a satisfactory answer in our previous reply. As discussed above, the non-Bloch PT symmetry is similar to the conventional PT symmetry, but defined for the non-Bloch Hamiltonian $H(k + i\kappa)$ that predicts the spectrum of a non-Hermitian Hamiltonian under OBCs. In other words, the necessary and sufficient conditions for the system to have non-Bloch PT symmetry is $[\sigma_x' \mathcal{K}, H(k + i\kappa)] = 0$ with σ_x' a two-by-two matrix satisfying $(\sigma_x' \mathcal{K})^2 = 1$.

Compared with PT symmetry, the non-Bloch PT symmetry is more sophisticated since one needs to first determine the explicit form of $H(k+i\kappa)$. Physically, κ describes the inverse localization length of skin modes, and can be obtained by requiring non-Bloch eigenenergies to form degenerate pairs for different quasi-momenta k (so that their linear combination can satisfy OBCs).

Explicitly in our model, $H(k+i\kappa)$ can be analytically obtained, and we find that the necessary and sufficient conditions for the system to be non-Bloch PT unbroken is $|u|<|\gamma/2|$. Otherwise, $H(k+i\kappa)$ becomes non-Bloch PT asymmetric (instead of broken). These properties, including a brief discussion of how to obtain $H(k+i\kappa)$, are discussed in the Method section (the section of "Non-Bloch PT symmetry").

Below, we further elaborate on how one can geometrically understand the conditions for yielding a real spectrum (non-Bloch PT symmetry) under OBCs, when the PBC spectrum is complex. This is based on the electrostatics analogy of the NHSE problem [7], which is beyond the context of this work. Nevertheless, it may lead to further extensions of our method based on this analogy, and we have now briefly mentioned this point in the discussion section. We also highlight the main points in this electrostatics analogy below:

In [7], an exact mapping was established between the behaviour of the energy eigenstates in the complex energy plane and the classical electrostatics problem in a physical setup of an analogous shape:

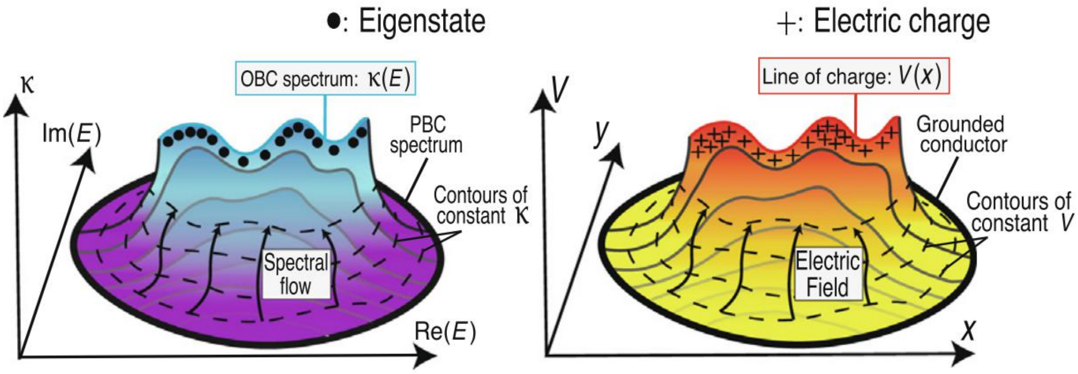

Fig 1(c,d) of [7]: the inverse skin localization depth κ corresponds to the electrical potential, such that the PBC spectrum (which by definition has vanishing κ) corresponds to a grounded conductor at zero electrical potential V. And the OBC spectrum corresponds to a charge density within the conductor, such that its skin depth κ corresponds to the potential of these charges. More explicitly, the mapping is as follows:

Non-Hermitian lattice	Electrostatic system	Remarks
Complex energy E^{a}	Complex position $z = x + iy^{a}$	
Inverse skin depth/ imaginary flux $\kappa(E)^{a,b}$	Electrical potential $V(z)^{a,b}$	Specified as input along the OBC spectrum
Density of states $\rho_{\epsilon}{}^{c}$	Electrostatic charge density $\sigma_{\epsilon}{}^{c}$	Nonzero along the OBC and PBC spectra
OBC to PBC spectral interpolation curve	Electric field lines of const. $U(z)$	
PBC spectrum $E(k)$	Grounded conductor $\epsilon^{PBC}(U)$	Specified as input
OBC spectrum $E(k + i\kappa)^{d}$	Path of nonzero charge density	Specified as input

The reason why this mapping holds is because the behaviour of the density of states in the NHSE problem can be shown to be equivalent to that of the spatial charge density subject to Poisson's equation in electrostatics. In [7], this is shown in detail, and fundamentally rests on the Cauchy-Riemann relations satisfied for the real and imaginary parts of the complex energy.

Now, let us answer the referee's question on the necessary and sufficient conditions for non-Bloch PT symmetry (having real OBC spectrum): Through this mapping, the latter is equivalent to having electrostatic charge all lying along a horizontal line (corresponding to the real energy axis), while being encircled by a conductor of a given shape (corresponding to the PBC spectrum for a given model). For a given conductor shape, which specifies the boundary conditions, we can place the desired charge density along the real line, and use a Poisson's PDE solver to solve for the corresponding potential V, which gives the κ for this charge density. We can also solve for the linear charge density induced on the conductor, whereby we can extract the PBC energy dispersion, as detailed in [7]. To conclude, in general, we can also find a PBC Hamiltonian whose OBC spectrum is given in a desired real window, but in many cases, this would involve an arbitrary non-local PBC Hamiltonian. If we allow only for sufficiently local PBC Hamiltonians, then whether a solution exists depends on whether the obtained energy dispersion can be satisfied by a sufficiently local PBC Hamiltonian.

Now, suppose we are already given a fixed PBC Hamiltonian, and wants to ask the question on whether a real OBC spectrum exists. This is more straightforward, because we can just set up the Poisson's equation solver and solve for the possible field lines in the interior of the (PBC) conductor. Here, we also have Neumann boundary conditions, because the given PBC Hamiltonian comes with a fixed dispersion, and that needs to be reflected in the charge density along the conductor. If there exists a field solution where we can have charges only along the real horizontal line, then we can have non-Bloch PT symmetry for the OBC spectrum.

In all, we see that the conditions for non-Bloch PT symmetry do not have a simple short answer except in the simplest models. However, upon reducing it to a numerical PDE problem, we see that, at least conceptually, the problem has a well-defined solution.

References:

[1] C. M. Bender, M. V. Berry, and A. Mandilara, Journal of Physics A: Mathematical and General 35, L467-L471 (2002).

[2] F. Song, H.-Y. Wang, and Z. Wang, in A Festschrift in Honor of the C. N. Yang Centenary, (2022), pp. 299–311.

[3] S. Yao, and Z. Wang, Phys. Rev. Lett. 121, 086803 (2018).

[4] C. H. Lee, L. Li, R. Thomale, and J. Gong, Phys. Rev. B 102, 085151 (2020).

[5] A. Mostafazadeh, Journal of Mathematical Physics 43, 205 (2002).

[6] K. Yokomizo and S. Murakami, Phys. Rev. Lett. 123, 066404 (2019).

[7] R. Yang et al., Sci. Bull. 67, 1865 (2022).

REVIEWERS' COMMENTS:

Reviewer #2 (Remarks to the Author):

We appreciate the authors for clarifying my concerns. They explain that the model investigated in the present work preserves the generalized PT symmetry depending on the system parameters. The utilization of the dependence allows us to control the values which the energy spectrum of the non-Hermitian SSH model takes, by adding the additional momentum ky. Therefore, in the demonstrated model, the rich localization of the bulk eigenstates can emerge under the single open boundary condition. I understand that the generalized PT symmetry can certainly switch the localization behaviors.

Meanwhile, the authors answer that, in general cases, it is difficult to provide the necessary and sufficient condition for the non-Bloch Hamiltonian taking the real-valued energy spectrum. However, they also explain that the above problem can be investigated based on the electrostatics. At least, one can numerically determine whether or not, for a given Bloch Hamiltonian, the non-Hermitian system takes the real-valued energy eigenvalue, which indicates that the proposed method allows us to effectively design the desirable model exhibiting the PT-activated non-Hermitian skin effect. I think that the seek of the general condition is beyond the work.

Thus, I would like to change my opinions and recommend publishing the manuscript for Communications Physics.