Supplementary Materials for "Resource Theory of Imaginarity in Distributed Scenarios"

SUPPLEMENTARY NOTE 1: PROOF OF THEOREM 2

In the following, we assume that A and B is a qubit. A general two-qubit state ρ^{AB} can be written as

$$\rho = \frac{1}{4}\left(\mathbb{1}\otimes\mathbb{1} + \sum_k a_k\sigma_k\otimes\mathbb{1} + \sum_l b_l\mathbb{1}\otimes\sigma_l + \sum_{k,l} E_{kl}\sigma_k\otimes\sigma_l\right), \tag{1}$$

where $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$ are local Bloch vectors of Alice and Bob, respectively, and $E_{kl} = \mathrm{Tr}(\sigma_k\otimes\sigma_l\rho)$. A general single-qubit POVM element on Alice's side can be written as

$$M_n^A = q_n\left(\mathbb{1} + \sum_j \alpha_{nj}\sigma_j\right) \tag{2}$$

with probabilities $0 \leq q_n \leq 1$, $\sum_n q_n = 1$, and vectors $\boldsymbol{\alpha}_n$ such that $|\boldsymbol{\alpha}_n| \leq 1$ and $\sum_n q_n\boldsymbol{\alpha}_n = 0$. The measurement $\{M_n^A\}$ gives outcome n with probability

$$p_n = q_n\left(1 + \boldsymbol{a}\cdot\boldsymbol{\alpha}_n\right), \tag{3}$$

and the Bloch vector of Bob's post-measurement state is

$$\boldsymbol{b}_n = \frac{\boldsymbol{b} + E^T\boldsymbol{\alpha}_n}{1 + \boldsymbol{a}\cdot\boldsymbol{\alpha}_n}. \tag{4}$$

After Alice communicates her measurement outcome n to Bob, he applies a real operation Λ_n to his post-measurement state ρ_i^B. For each measurement outcome n, Bob aims to maximize the fidelity between $\Lambda_n[\rho_n^B]$ and the maximally imaginary state $|\hat{+}\rangle$. The maximum is given by the fidelity of imaginarity F_I which for single-qubit states ρ_n^B reduces to

$$F_\mathrm{I}(\rho_n^B) = \frac{1}{2}\left(1 + \left|\mathrm{Tr}[\rho_n^B\sigma_2]\right|\right). \tag{5}$$

Using this result together with Eqs. (3) and (4) we can express our figure of merit F_a as follows:

$$\begin{aligned} F_\mathrm{a}(\rho^{AB}) &= \max_{M_n^A}\sum_n p_n F_\mathrm{I}(\rho_i^B) \\ &= \max_{q_n,\boldsymbol{\alpha}_n}\frac{1}{2}\left(1 + \sum_n q_n\left|b_2 + \boldsymbol{s}\cdot\boldsymbol{\alpha}_n\right|\right), \end{aligned} \tag{6}$$

where the maximization in the last expression is performed over all vectors $\boldsymbol{\alpha}_n$ and probabilities $0 \leq q_n \leq 1$ such that $\sum_n q_n = 1$, $|\boldsymbol{\alpha}_n| \leq 1$ and $\sum_n q_n\boldsymbol{\alpha}_n = 0$.

If $|b_2| \geq |\boldsymbol{s}|$, then using the conditions $|\boldsymbol{\alpha}_n| \leq 1$ and $\sum_n q_n\boldsymbol{\alpha}_n = 0$ we immediately obtain

$$\sum_n q_n\left|b_2 + \boldsymbol{s}\cdot\boldsymbol{\alpha}_n\right| = |b_2| \tag{7}$$

for any choice of q_n and $\boldsymbol{\alpha}_n$. This directly implies that $F_\mathrm{a}(\rho^{AB}) = 1/2 + |b_2|/2$ in this case, in accordance with $F_\mathrm{a}\left(\rho^{AB}\right) = \frac{1}{2}\left(1 + \max\left\{|b_2|, |\boldsymbol{s}|\right\}\right)$.

We now consider the case if $|b_2| < |\boldsymbol{s}|$. We will show that in the maximization in Eq. (6) it is enough to consider POVMs consisting of two elements. For a given set of vectors $\boldsymbol{\alpha}_n$ and probabilities q_n we introduce two sets, depending whether $b_2 + \boldsymbol{s}\cdot\boldsymbol{\alpha}_n$ is positive or negative:

$$S_0 = \{n : b_2 + \boldsymbol{s}\cdot\boldsymbol{\alpha}_n \geq 0\}, \tag{8a}$$
$$S_1 = \{j : b_2 + \boldsymbol{s}\cdot\boldsymbol{\alpha}_j < 0\}. \tag{8b}$$

Using these sets, we express the sum $\sum_n q_n\left|b_2 + \boldsymbol{s}\cdot\boldsymbol{\alpha}_n\right|$ as follows:

$$\begin{aligned} \sum_n q_n\left|b_2 + \boldsymbol{s}\cdot\boldsymbol{\alpha}_n\right| = &\left(\sum_{n\in S_0} q_n\right)\left|b_2 + \frac{\sum_{n\in S_0} q_n(\boldsymbol{s}\cdot\boldsymbol{\alpha_n})}{\sum_{n\in S_0} q_n}\right| \\ &+ \left(\sum_{j\in S_1} q_j\right)\left|b_2 + \frac{\sum_{j\in S_1} q_j(\boldsymbol{s}\cdot\boldsymbol{\alpha_j})}{\sum_{j\in S_1} q_j}\right|. \end{aligned} \tag{9}$$

In the next step, we introduce the probabilities $\tilde{q}_0 = \sum_{n\in S_0} q_n$, $\tilde{q}_1 = \sum_{j\in S_1} q_j$ and vectors

$$\tilde{\boldsymbol{\alpha}}_0 = \frac{\sum_{n\in S_0} q_n\boldsymbol{\alpha}_n}{\sum_{n\in S_0} q_n}, \tag{10a}$$
$$\tilde{\boldsymbol{\alpha}}_1 = \frac{\sum_{j\in S_1} q_j\boldsymbol{\alpha}_j}{\sum_{j\in S_j} q_j}. \tag{10b}$$

Noting that

$$b_2 + \boldsymbol{s}\cdot\tilde{\boldsymbol{\alpha}}_0 \geq 0, \tag{11a}$$
$$b_2 + \boldsymbol{s}\cdot\tilde{\boldsymbol{\alpha}}_1 < 0, \tag{11b}$$

we further obtain the following result:

$$\begin{aligned} \sum_n q_n\left|b_2 + \boldsymbol{s}\cdot\boldsymbol{\alpha}_n\right| &= \tilde{q}_0|b_2 + \boldsymbol{s}\cdot\tilde{\boldsymbol{\alpha}}_0| + \tilde{q}_1|b_2 + \boldsymbol{s}\cdot\tilde{\boldsymbol{\alpha}}_1| \\ &= \tilde{q}_0(b_2 + \boldsymbol{s}\cdot\tilde{\boldsymbol{\alpha}}_0) - \tilde{q}_1(b_2 + \boldsymbol{s}\cdot\tilde{\boldsymbol{\alpha}}_1). \end{aligned} \tag{12}$$

The vectors $\tilde{\boldsymbol{\alpha}}_n$ and probabilities $\tilde{q}_n$ fulfill the conditions $\sum_n\tilde{q}_n = 1$, $|\tilde{\boldsymbol{\alpha}}_n| \leq 1$, and $\sum_n\tilde{q}_n\tilde{\boldsymbol{\alpha}}_n = 0$. This implies that they correspond to a two-element POVM on Alice's side via the relation in Eq. (2).

The arguments just presented show that the maximum in Eq. (6) can be achieved with two vectors $\boldsymbol{\alpha}_0$ and $\boldsymbol{\alpha}_1$ and two probabilities q_0 and q_1 having the properties $0 \leq q_0 \leq 1$, $q_1 = 1 - q_0$, $|\boldsymbol{\alpha}_n| \leq 1$, $\sum_i q_n\boldsymbol{\alpha}_n = 0$. To complete the proof, we will show that the optimal solution is obtained for

$$q_0 = q_1 = \frac{1}{2}, \tag{13a}$$

$$\boldsymbol{\alpha}_0 = -\boldsymbol{\alpha}_1 = \frac{\boldsymbol{s}}{|\boldsymbol{s}|}. \tag{13b}$$

Recalling that $|b_2| \leq |\boldsymbol{s}|$, the values in Eq. (13) immediately give a lower bound on the assisted fidelity of imaginarity:

$$F_{\mathrm{a}}(\rho^{AB}) \geq \frac{1}{2}(1 + |\boldsymbol{s}|). \tag{14}$$

Let now q_n and $\boldsymbol{\alpha}_n$ be optimal probabilities and vectors [not necessarily coinciding with Eq. (13)]. Without loss of generality we can assume that

$$b_2 + \boldsymbol{s}\cdot\boldsymbol{\alpha}_0 \geq 0, \tag{15a}$$

$$b_2 + \boldsymbol{s}\cdot\boldsymbol{\alpha}_1 < 0. \tag{15b}$$

Otherwise, if $b_2 + \boldsymbol{s}\cdot\boldsymbol{\alpha}_n$ is positive (or negative) for all n, we obtain $\sum_n q_n |b_2 + \boldsymbol{s}\cdot\boldsymbol{\alpha}_n| = |b_2|$. Since $|b_2| < |\boldsymbol{s}|$, this means that we will not be able to reach the maximal value.

For the assisted fidelity of imaginarity we thus obtain

$$F_{\mathrm{a}}(\rho^{AB}) = \frac{1}{2}[q_0(b_2 + \boldsymbol{s}\cdot\boldsymbol{\alpha}_0) - q_1(b_2 + \boldsymbol{s}\cdot\boldsymbol{\alpha}_1)] + \frac{1}{2}. \tag{16}$$

Since $q_0 + q_1 = 1$, it must be that either $q_0 \leq 1/2$ or $q_1 \leq 1/2$. In the first case we rewrite Eq. (16) as follows:

$$F_{\mathrm{a}}(\rho^{AB}) = \frac{1-b_2}{2} + q_0(b_2 + \boldsymbol{s}\cdot\boldsymbol{\alpha}_0) \leq \frac{1}{2}(1 + |\boldsymbol{s}|). \tag{17}$$

In the second case ($q_1 \leq 1/2$), we rewrite Eq. (16) as

$$F_{\mathrm{a}}(\rho^{AB}) = \frac{1+b_2}{2} - q_1(b_2 + \boldsymbol{s}\cdot\boldsymbol{\alpha}_1) \leq \frac{1}{2}(1 + |\boldsymbol{s}|). \tag{18}$$

Thus, for $|b_2| < |\boldsymbol{s}|$ the assisted fidelity of imaginarity is bounded above as

$$F_{\mathrm{a}}(\rho^{AB}) \leq \frac{1}{2}(1 + |\boldsymbol{s}|). \tag{19}$$

Together with Eq. (14) this proves that $F_{\mathrm{a}}(\rho^{AB}) = 1/2 + |\boldsymbol{s}|/2$ in this case, and the proof of the theorem is complete.

SUPPLEMENTARY NOTE 2: PROOF OF LEMMA 3

Note that, the geometric measure of imaginarity and the concurrence of imaginarity are given by

$$G(\rho) = \min_e \sum_j p_j \frac{1 - |\langle\psi_j^*|\psi_j\rangle|}{2} = \frac{1 - \sqrt{F(\rho,\rho^T)}}{2} \tag{20}$$

$$C(\rho) = \min_e \sum p_j |\langle\psi_j^*|\psi_j\rangle| = \max\left\{0, \lambda_1 - \sum_{k>1}\lambda_k\right\} \tag{21}$$

In the above $\max_e$ and $\min_e$ are maximisation and minimisation over pure state ensembles of ρ. Whereas, $\{\lambda_1, \lambda_2 ...\}$ are the eigenvalues (in decreasing order) of $(\sqrt{\rho}\rho^T\sqrt{\rho})^{\frac{1}{2}}$. In general, for probabilistic transformations, the following inequality holds

$$p(\rho \to \sigma) \leq \min\left\{\frac{G(\rho)}{G(\sigma)}, 1\right\}. \tag{22}$$

It was further shown in [?], that the optimal probability of converting a pure state ψ to a arbitrary quantum state ρ is given by

$$p(\psi \to \rho) = \min\left\{\frac{G(\psi)}{G(\rho)}, 1\right\}. \tag{23}$$

In a one way LQRCC procedure, Alice performs a general quantum measurement and corresponding to the outcomes (with probabilites $\{p_j\}$) of Alice, Bob's local state is found in the state ρ_j, such that, $\{p_j, \rho_i\}$ is an ensemble of ρ^B. Conditioned on the outcome of Alice (i), Bob can perform a local stochastic real operation on ρ_i, probabilistically converting it into σ^B. Using Eq. (22) and Eq. (23), it follows

$$P_a \leq \sum_j p_j \min\left\{\frac{G(\rho_j)}{G(\sigma_B)}, 1\right\} \leq \sum_{jk} p_j q_k \min\left\{\frac{G(\psi_{j,k})}{G(\sigma_B)}, 1\right\}. \tag{24}$$

The second inequality follows from Eq.(20), $G(\rho_j)$ is calculated by minimising over all pure state ensembles of ρ_j. Therefore, the second inequality holds for any pure state decomposition of ρ_j, like $\{q_k, \psi_{jk}\}$. Note that $\{p_j q_k, \psi_{jk}\}$ is a pure state decomposition of ρ^B. Note that, any pure state decomposition of ρ^B can be realised by a suitable local measurement by Alice. Using this fact, along with Eq.(20) and Eq.(23) implies that

$$\begin{aligned} P_a &= \min\left\{\frac{1 - \min_e \sum_k p_k |\langle\psi_k|\psi_k^*\rangle|}{2G(\sigma^B)}, 1\right\} \\ &= \min\left\{\frac{1 - C(\rho^B)}{1 - \sqrt{F(\sigma^B, (\sigma^B)^T)}}, 1\right\}. \end{aligned}$$

Here, $\min_e$ is the minimisation over pure state ensembles of ρ^B. This completes the proof.

SUPPLEMENTARY NOTE 3: PROOF OF THEOREM 4

From Lemma 1, we know that optimal probability for Bob to locally achieve σ^B from a shared bipartite pure state ψ^{AB} with unit fidelity, via LQRCC is given by

$$P(\psi^{AB} \to \sigma^B) = \min\left\{\frac{1 - \mathfrak{I}_c(\rho^B)}{2\mathfrak{I}_g(\rho)}, 1\right\}. \tag{25}$$

If we want to achieve σ^B with fidelity at least f, the best strategy is to go to a state (σ'^B), within the fidelity ball around σ^B, with a minimal geometric measure of imaginarity. Therefore,

$$P_f(\psi^{AB} \to \sigma^B) = \min\left\{\frac{1-\mathcal{I}_c(\rho^B)}{2\mathcal{I}_g(\sigma'^B)}, 1\right\}. \quad (26)$$

From [**?**], we know that

$$\mathcal{I}_g(\sigma'^B) = \sin^2\left(\max\left\{\sin^{-1}\sqrt{\mathcal{I}_g(\sigma^B)} - \cos^{-1}\sqrt{f}, 0\right\}\right). \quad (27)$$

We now define

$$m = \sin^{-1}\sqrt{\frac{1-\mathcal{I}_c(\rho^B)}{2}} - \sin^{-1}\sqrt{\mathcal{I}_g(\sigma^B)} + \cos^{-1}\sqrt{f}. \quad (28)$$

First, consider the case when $m \geq 0$, which implies

$$\sin^{-1}\sqrt{\mathcal{I}_g(\sigma^B)} - \cos^{-1}\sqrt{f} \leq \sin^{-1}\sqrt{\frac{1-\mathcal{I}_c(\rho^B)}{2}}. \quad (29)$$

We know that

$$\sin^{-1}\sqrt{\mathcal{I}_g(\sigma^B)} - \cos^{-1}\sqrt{f} \in [-\pi/2, \pi/4] \quad (30)$$

and $\sin^{-1}\sqrt{\frac{1-\mathcal{I}_c(\rho^B)}{2}} \in [0, \pi/4]$. Therefore,

$$\max\left\{\sin^{-1}\sqrt{\mathcal{I}_g(\sigma^B)} - \cos^{-1}\sqrt{f}, 0\right\} \leq \sin^{-1}\sqrt{\frac{1-\mathcal{I}_c(\rho^B)}{2}}. \quad (31)$$

Using these results, we get

$$\begin{aligned}\mathcal{I}_g(\sigma^{B'}) &= \sin^2\left(\max\left\{\sin^{-1}\sqrt{\mathcal{I}_g(\sigma^B)} - \cos^{-1}\sqrt{f}, 0\right\}\right) \quad (32)\\ &\leq \sin^2\left(\sin^{-1}\sqrt{\frac{1-\mathcal{I}_c(\rho^B)}{2}}\right) = \frac{1-\mathcal{I}_c(\rho^B)}{2}.\end{aligned}$$

For the case when $\frac{1-\mathcal{I}_c(\rho^B)}{2} > 0$, the above inequality implies

$$\frac{1-\mathcal{I}_c(\rho^B)}{2\mathcal{I}_g(\sigma^{B'})} \geq 1. \quad (33)$$

This shows that $P_f(\psi^{AB} \to \sigma^B) = 1$ when $m \geq 0$.

Now, we look at the other case when $m < 0$, i.e.,

$$\sin^{-1}\sqrt{\mathcal{I}_g(\sigma^B)} - \cos^{-1}\sqrt{f} > \sin^{-1}\sqrt{\frac{1-\mathcal{I}_c(\rho^B)}{2}} > 0. \quad (34)$$

From the above inequality and Lemma 1, we have

$$P_f(\psi^{AB} \to \sigma^B) = \frac{1-\mathcal{I}_c(\rho^B)}{2\sin^2(\sin^{-1}\sqrt{\mathcal{I}_g(\sigma^B)} - \cos^{-1}\sqrt{f})}. \quad (35)$$

Using the above result, a closed expression can also be found for F_p. Let's first consider the case when $p \leq \frac{1-\mathcal{I}_c(\psi^{AB})}{2\mathcal{I}_g(\sigma^B)} < 1$, in this case $F_p(\psi \to \sigma^B) = 1$ (follows from Lemma 1). When $1 \geq p > \frac{\mathcal{I}_g(\psi)}{\mathcal{I}_g(\sigma^B)}$, the optimal achievable fidelity can be obtained by solving Eq. (**??**) for f, which gives

$$F_p(\psi^{AB} \to \sigma^B) = \cos^2\left[\sin^{-1}\sqrt{\mathcal{I}_g(\sigma^B)} - \sin^{-1}\sqrt{\frac{1-\mathcal{I}_c(\rho^B)}{2p}}\right].$$

This completes the proof.