

Boundary-localized many-body bound states in the continuum



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Reviewers' comments:

Reviewer #1 (Remarks to the Author):

This manuscript presents a theoretical and experimental investigation on interaction-induced many-body bound states in the continuum, which can be perfectly localized at the boundaries of a finite 1D lattice without requiring any defective potential. Interestingly, the authors have demonstrated that these boundary-localized multi-boson states do not couple with extended bulk modes, even when their eigenenergies are mixed. Unlike previously discovered single- or many-body BICs that rely on carefully designed coupling configurations, defective potentials or nonlocal interactions, this new type of boundary many-body BIC arises solely from Hubbard interactions, providing a novel mechanism for designing BICs. In experiments, the authors have mapped Fock states of three and four bosons to eigenmodes of nonlinear circuit networks and successfully simulated these boundary many-body BICs. The realization of many-body boundary BICs is highly intriguing, and the experimental simulation using nonlinear circuits is also well-executed. I strongly recommend this paper for publication in Communications Physics once the following comments are addressed.

- 1) The authors have employed two approaches to demonstrate the existence of boundary many-body bound states in the finite 1D lattice depicted in Fig. 1d and 1e. Can these two methods unveil distinct properties, or do they consistently manifest the perfectly localized nature of boundary many-body bound states?
- 2) Fig. 2e has shown that the interaction threshold of multi-boson BICs decreases with the number of bosons being increased. Some discussions on the origin of this phenomenon should be added.
- 3) Could the effective interaction strength be adjustable in the nonlinear circuit? Additionally, how does the imperfect performance of active circuit elements and disorder effects impact the experimental simulation of boundary many-body BICs?
- 4) Lastly, BICs have been widely realized in various classical-wave systems, possessing important applications such as low-threshold lasers. It is useful to provide the discussion on the potential applications of BICs in electric circuits.
- 5) Authors should demonstrate the equivalence between the nonlinear circuit networks and interaction-induced many-body bound states in the continuum.

Reviewer #2 (Remarks to the Author):

The authors theoretically predicted and experimentally demonstrated the existence of many-body bound state in the continuum (BIC) in the well-known Bose Hubbard model. The many-body BICs are localized at the boundaries as a result of uniform onsite interaction, without requiring impurity potential or specific design of interactions. The many-body BIC can be tuned out of the continuum by increasing the interaction strength. By mapping the couplings of Fock-state basis to nonlinear circuit networks, the authors prepared and showed the density distribution of many-body BICs in electric circuits.

The generalization of BIC from single-body quantum systems or classical-wave systems to many-body quantum systems have attracted great interest and attentions in recent years. Although there are some mechanisms that can faithfully produce many-body BICs, it is always appealing to find new mechanisms for many-body BICs. Compared with previous works, this work has showed an interesting mechanism for many-body BICs, that is, particle-particle interaction and boundary effect together give rise to many-body boundary BICs. This manuscript is timely, well organized and friendly to readers. The results are fresh, attracted and interesting. However, before further evaluation, the authors should address the following comments and suggestions as follows.

#1 The correlations of the many-body BICs are lack. Although we know the density distribution of the many-body BICs, we are unclear about the correlations between particles, which are as important as the density distribution. I suggestions the authors give second-order and third-order correlation functions for the three-body BICs.

#2 Although the authors tried to give mechanism for many-body BICs in the supplementary information, their descriptions are still rough and cannot well explain the basic mechanism. I believe both the onsite interaction and the boundary effect play important role in the formation of many-body boundary BICs. In the periodic boundary condition, there should be no such many-body BIC at all. The authors emphasize this point in some place of their main text. Furthermore, It would be better for the authors to give clearer explanation of the mechanism.

#3 It seems that such many-body BICs exist in systems with as least three particles. Could the authors explain why two-particle BICs are absent?

#4 In the caption of Figure 1, the authors choose some special states to present their results. Could the authors explain why they choose the states with indices “ 4302, 22051, 22097 and 21934”?

#5 In Eq. (4), there is a strange number “7.6”. In the supplementary information, “7.6” suddenly appear in Eq. (S18), before which there is no such number. Where this number comes from? What parameters are this number related to?

#6 How the number of many-body boundary BICs increases with the number of particles?

#7 In Figure 3, the experimental results depart quite a lot from the theoretical results. The authors explain that the deviation arises from loss effects in the circuit sample. Could the authors also give numerical results that take the loss effects into account?

#8 The system size is quite small in the experiments. It is hard to claim the state in Figure 3(d1) as BIC with such small systems. Could the authors present experimental results of three particles in six lattices or more?

#9 The authors may give a section in the supplementary information to explain how and why they can excite the BICs in experiments since they are lying in the continuum spectrum.

#10 There are some typos in the main text and supplementary information, such as “these two five-boson bound states are boundry BICs”, “ (a) and (b) present two-boson and three-boson Fock spaces”. The authors should carefully check their manuscripts.

Reviewer #3 (Remarks to the Author):

In the manuscript entitled “Boundary many-body bound states in the continuum”, the authors propose a standard Bose-Hubbard simple model in a homogeneous one-dimensional lattice, where three or more interacting bosons can concentrate on the boundary and the eigenenergy of multi-boson bound states can embed within the continuous energy spectra of extended scattering states. The authors further experimentally simulate the many-body BICs based on a mapping to nonlinear circuit networks. The novelty of this work is that it provides a new mechanism to realize many-body BIC and implements it in nonlinear electric circuits, which enables it to be published in Communications Physics. There are some concerns about the manuscript at this current stage.

Main concerns:

1. A theoretical comparison in detail with Refs 35 and 36 will be appreciated for the readers who are not familiar with the many-body BICs. Moreover, the statement “the two-BASs with interaction U (two blue shaded areas) emulate the effect of the defect potential of three-BASs with the interaction strength being $3U$ (red shaded areas)” in Supplementary note 4 is not very clarified (similar statement also appear in the main text).

2. It may be not proper to write the single-PSS in Eq. (3) as $(\dots 0, 1, 1, 1, 0 \dots)$ since it excludes terms like $(\dots 1, 0, 1, 1, 0 \dots)$. Similar issues for the two-BASs and three-BASs in SI. Furthermore, what is the feature difference for the IPRs of $(\dots 0, 1, 1, 1, 0 \dots)$ and $(\dots 1, 0, 1, 1, 0 \dots)$, as well as of other single-PSS configuration? and for two-BASs and three-BASs?

3. As the authors follows “other numerical method on the demonstration of BICs [40, 41]” to plot Fig. 1(e) with non-Hermitian on-site terms, but there is no ultra deep profile or divergence tendency in Fig. 1(e). Why?

4. How to determine the threshold of $U=1.92$? It seems to be very hard to distinguish between BIC and simple boundary state, both of which share the same physical origin. In this regard, why are these specific three-BASs the BICs?

5. When comparing the results of three, four, and five-boson cases, is there a threshold of particle number to observe the boundary BIC under the same U ?

6. The mapping of the nonlinear electric circuits to the one-dimensional lattice model is very difficult to grasp or understand. The lattice model is clear, but how does the mapping work? Moreover, the role of nonlinearity seems important and needs further clarification.

Some typos:

(1) In Line 34, it should be [2, 3]. From Line 116 to 121, it should be Fig. 1 not just Fig.

(2) The caption in Supplementary Figure 5 should be for three-boson states.

Response Letter to Reviewers

We are grateful for the constructive comments on this manuscript (COMMSPHYS-24-0556-T) from three reviewers.

In the text below, reviewer comments are quoted in blue and followed by our detailed response. We have also revised the manuscript and the Supplemental Information based on the reviewer's comments, and these updates are highlighted in red in those files. In the text below, these updates are also highlighted in *Italics*.

Response to comments of the reviewer #1

This manuscript presents a theoretical and experimental investigation on interaction-induced many-body bound states in the continuum, which can be perfectly localized at the boundaries of a finite 1D lattice without requiring any defective potential. Interestingly, the authors have demonstrated that these boundary-localized multi-boson states do not couple with extended bulk modes, even when their eigenenergies are mixed. Unlike previously discovered single- or many-body BICs that rely on carefully designed coupling configurations, defective potentials or nonlocal interactions, this new type of boundary many-body BIC arises solely from Hubbard interactions, providing a novel mechanism for designing BICs. In experiments, the authors have mapped Fock states of three and four bosons to eigenmodes of nonlinear circuit networks and successfully simulated these boundary many-body BICs. The realization of many-body boundary BICs is highly intriguing, and the experimental simulation using nonlinear circuits is also well-executed. I strongly recommend this paper for publication in Communications Physics once the following comments are addressed.

Reply: We would like to thank the reviewer for the careful review and valuable suggestions of our work. We also appreciate the reviewer's positive comment on our theoretical/experimental achievements on many-body BICs, and the recommendation for publication. However, we note that the reviewer has also raised some concerns on our work. In the following, we provide detailed explanations on the correspondence between nonlinear circuit networks and interaction-induced BICs, numerical methods for determining BICs, and their potential applications. Furthermore, we have added these related discussions and results in the revised manuscript. We believe that the

reviewer will find the revised version of our manuscript considerably improved.

1. The authors have employed two approaches to demonstrate the existence of boundary many-body bound states in the finite 1D lattice depicted in Fig. 1d and 1e. Can these two methods unveil distinct properties, or do they consistently manifest the perfectly localized nature of boundary many-body bound states?

Reply: We would like to thank the reviewer for the comment. In fact, these two approaches can consistently manifest the perfectly localized nature of many-body BICs. In Fig. 1d, we numerically calculate the variation of IPR of all eigenmodes as a function of the lattice length. It is found that $IPRs$ of two eigenmodes converge to a saturated value with the lattice size being increased, manifesting the perfect localization properties. In addition, in Fig. 1e, we calculate the variation of the imaginary part of eigen-energies as a function of the size with artificial losses n_l . The exponential decrease in the imaginary components of eigen-energies appear. This phenomenon can also illustrate the perfect localization of three-boson BICs.

Action taken:

- In the revised manuscript, we have added the following discussion on page 7 of to clarify the consistence of two methods used in Figs. 1d and 1e: “*Thus, two approaches employed in Figs. 1(e) and 1(f) can consistently manifest the perfectly localized nature of boundary many-body bound states.*”.

2. Fig. 2e has shown that the interaction threshold of multi-boson BICs decreases with the number of bosons being increased. Some discussions on the origin of this phenomenon should be added.

Reply: We would like to thank the reviewer for the comment. The multi-boson BICs correspond to the state where all bosons are localized around the boundary of the lattice. In this case, the eigen-energy of a multi-boson BIC is directly proportional to the particle number N and interaction strength U as $\varepsilon_{BIC} \sim \frac{N(N-1)U}{2}$. Thus, as the particle number being increased, the threshold of interaction U needs to decrease to ensure that the eigen-energy of many-body BICs can still remain within the continuum spectra, which possess the lower growth rate than that of many-body BICs.

Action taken:

- In the page 9 of the revised manuscript, we have added the following discussion to illustrate the origin for the decreased interaction threshold: *“This is due to the fact that the eigen-energy of a multi-boson BIC is directly proportional to the particle number N and interaction strength U as $\varepsilon_{BIC} \sim N(N-1)U/2$. Thus, as the particle number being increased, the threshold of interaction U needs to decrease to ensure that the eigen-energy of many-body BICs can still remain within the continuum spectra, which possess the lower growth rate than that of many-body BICs.”*.

3. Could the effective interaction strength be adjustable in the nonlinear circuit? Additionally, how does the imperfect performance of active circuit elements and disorder effects impact the experimental simulation of boundary many-body BICs?

Reply: We would like to thank the reviewer for the comment. Using our designed nonlinear circuits, the effective interaction strength can be suitably adjusted by tuning circuit parameters. Specifically, we can control the amplification factor of the multiplier (through R_1 and R_2 shown in Fig. S4 in the Supplementary Information) to adjust the effective interaction strength. This allows us to simulate few-boson BICs with different effective interaction strengths. Additionally, the imperfect performance of active circuit elements and disorder effects can broaden the measured impedance peaks, thereby reducing the localization strength of boundary many-body BICs.

Action taken:

- In page 10 of the revised manuscript, we have added the following discussion to clarify the tunable interaction strength: *“Specifically, we can control the amplification factor of the multiplier (through R_1 and R_2 shown in Fig. 3(a)) to adjust the effective interaction strength. This allows us to simulate few-boson BICs with different effective interaction strengths.”*.
- In page 13 of the revised manuscript, we have added the following discussion to clarify the imperfect performance of nonlinear circuits: *“The lower localization strength of the measured BIC can be attributed to the loss effect, arising from imperfect performance of active circuit elements and disorder effects within the circuit sample, which leads to broadening of the measured impedance peaks.”*.

4. Lastly, BICs have been widely realized in various classical-wave systems, possessing important

applications such as low-threshold lasers. It is useful to provide the discussion on the potential applications of BICs in electric circuits.

Reply: We would like to thank the reviewer for the comment. Inspired by the potential applications of BICs in other classical-wave systems, we expect that BICs in electric circuits could possess potential applications in the fields of electronic sensors with high-sensitivity. Moreover, nonlinear circuits are also able to be used as the classical simulator to investigate other quantum few-body phenomena.

Action taken:

- In page 14 of the revised manuscript, we have added the following discussion to illustrate potential applications of BICs in electric circuits: *“Inspired by potential applications of BICs in other classical-wave systems, we expect that BICs in electric circuits could possess potential applications in the fields of electric sensors with high-sensitivity. Moreover, nonlinear circuits are also able to be used as the classical simulators to investigate other quantum few-body phenomena.”*.

5. Authors should demonstrate the equivalence between the nonlinear circuit networks and interaction-induced many-body bound states in the continuum.

Reply: We would like to thank the reviewer for the comment. The correspondence between eigen-equations of nonlinear circuits and many-body lattice models can demonstrate the efficacy of simulating many-body BICs using nonlinear circuits. Specifically, as for the three-boson lattice model, the corresponding eigen-equations with respect to three representative basis vectors, which are $|\dots 0,1,1,1,0 \dots\rangle$, $|\dots 0,2,1,0 \dots\rangle$, and $|\dots 0,3,0,0 \dots\rangle$, in three-boson Fock space can be expressed as:

$$\varepsilon c_{|\dots 0,1,1,1,0 \dots\rangle} = -t(c_{|\dots 1,0,1,1,0 \dots\rangle} + \sqrt{2}c_{|\dots 0,0,2,1,0 \dots\rangle} + \sqrt{2}c_{|\dots 0,2,0,1,0 \dots\rangle} + \sqrt{2}c_{|\dots 0,1,0,2,0 \dots\rangle} + \sqrt{2}c_{|\dots 0,1,2,0,0 \dots\rangle} + c_{|\dots 0,1,1,0,1 \dots\rangle}), \quad (\text{R1})$$

$$\varepsilon c_{|\dots 0,2,1,0 \dots\rangle} = -t(\sqrt{2}c_{|\dots 1,1,1,0 \dots\rangle} + 2c_{|\dots 0,1,2,0 \dots\rangle} + c_{|\dots 0,2,0,1 \dots\rangle} + \sqrt{3}c_{|\dots 0,3,0,0 \dots\rangle}) + U, \quad (\text{R2})$$

$$\varepsilon c_{|\dots 0,3,0,0 \dots\rangle} = -t(\sqrt{3}c_{|\dots 1,2,0,0 \dots\rangle} + \sqrt{3}c_{|\dots 0,2,1,0 \dots\rangle}) + 3U. \quad (\text{R3})$$

where $c_{|n_1, n_2, \dots, n_L\rangle}$ is the probability amplitude of finding n_i particle at the i th site for the N -boson state. As for the designed nonlinear circuit, we can obtain the eigenequations at different circuit nodes based on the Kirchhoff's law. It is noted that the amount of circuit nodes is consistent with that of three-boson basis vectors in Fock space, where the voltage $V_{|n_1, n_2, \dots, n_L\rangle}$ at the circuit node

labeled by $|n_1, n_2, \dots, n_L\rangle$ is mapped to the probability amplitude of $c_{|n_1, n_2, \dots, n_L\rangle}$. Thus, circuit eigenequations with respect to three circuit nodes with $V_{|\dots 0, 1, 1, 1, 0 \dots\rangle}$, $V_{|\dots 0, 2, 1, 0 \dots\rangle}$ and $V_{|\dots 0, 3, 0, 0 \dots\rangle}$ can be expressed as:

$$(f_0^2/f^2 - 7.6)V_{|\dots 0, 1, 1, 1, 0 \dots\rangle} = -(V_{|\dots 1, 0, 1, 1, 0 \dots\rangle} + \sqrt{2}V_{|\dots 0, 2, 1, 0 \dots\rangle} + \sqrt{2}V_{|\dots 0, 2, 0, 1, 0 \dots\rangle} + \sqrt{2}V_{|\dots 0, 1, 0, 2, 0 \dots\rangle} + \sqrt{2}V_{|\dots 0, 1, 2, 0, 0 \dots\rangle} + V_{|\dots 0, 1, 1, 0, 1 \dots\rangle}) + U_{single}V_{|\dots 0, 1, 1, 1, 0 \dots\rangle}, \quad (R4)$$

$$(f_0^2/f^2 - 7.6)V_{|\dots 0, 2, 1, 0 \dots\rangle} = -(\sqrt{2}V_{|\dots 1, 1, 1, 0 \dots\rangle} + 2V_{|\dots 0, 1, 2, 0 \dots\rangle} + V_{|\dots 0, 2, 0, 1 \dots\rangle} + \sqrt{3}V_{|\dots 0, 3, 0, 0 \dots\rangle}) + U_{two}V_{|\dots 0, 2, 1, 0 \dots\rangle}, \quad (R5)$$

$$(f_0^2/f^2 - 7.6)V_{|\dots 0, 3, 0, 0 \dots\rangle} = -(\sqrt{3}V_{|\dots 1, 2, 0, 0 \dots\rangle} + \sqrt{3}V_{|\dots 0, 2, 1, 0 \dots\rangle}) + U_{three}V_{|\dots 0, 3, 0, 0 \dots\rangle}, \quad (R6)$$

where U_{single} , U_{two} and U_{three} are proportional to the amplification factors of multipliers used to simulate effective interaction strengths of single-particle scattering states, two-boson and three-boson aggregative states. In addition, the eigen-energy (ε) of the three-boson lattice model is directly mapped to eigen-frequency (f) of the nonlinear circuit with $\varepsilon = f_0^2/f^2 - 7.6$. The detailed deviation of Eqs. (R4)- (R6) is provided in [Supplementary Note 6](#), elucidating how the relationship $\varepsilon = f_0^2/f^2 - 7.6$ can be derived.

Comparing Eqs. (R1)- (R3) and Eqs. (R4)- (R6), it is evident that the eigen-equations of our designed nonlinear circuits and many-body lattice model exhibit the identical form, implying the effectiveness of using nonlinear circuits for simulating many-body BICs in the lattice model. It is noted that the eigenequations with respect to other basis vectors in Fock space also possess the same form to that of other corresponding circuit nodes.

Action taken:

- In page 12 of the revised manuscript, we have added the following discussion to emphasize the equivalence between the nonlinear circuit networks and interaction-induced many-body bound states in the continuum: *“It is shown that the eigen-equations with respect to circuit nodes mapped to all basis vectors in Fock space possess the same form, implying the effectiveness of using nonlinear circuits for simulating many-body BICs in the lattice model.”*.
- In Supplementary Note 6, we have provided the detailed deviation and discussion to illustrate the equivalence between the nonlinear circuit networks and interaction-induced many-body bound states in the continuum.

Response to comments of the reviewer #2

The authors theoretically predicted and experimentally demonstrated the existence of many-body bound state in the continuum (BIC) in the well-known Bose Hubbard model. The many-body BICs are localized at the boundaries as a result of uniform onsite interaction, without requiring impurity potential or specific design of interactions. The many-body BIC can be tuned out of the continuum by increasing the interaction strength. By mapping the couplings of Fock-state basis to nonlinear circuit networks, the authors prepared and showed the density distribution of many-body BICs in electric circuits.

The generalization of BIC from single-body quantum systems or classical-wave systems to many-body quantum systems have attracted great interest and attentions in recent years. Although there are some mechanisms that can faithfully produce many-body BICs, it is always appealing to find new mechanisms for many-body BICs. Compared with previous works, this work has showed an interesting mechanism for many-body BICs, that is, particle-particle interaction and boundary effect together give rise to many-body boundary BICs. This manuscript is timely, well organized and friendly to readers. The results are fresh, attracted and interesting. However, before further evaluation, the authors should address the following comments and suggestions as follows.

Reply: We would like to thank the reviewer for the careful review and valuable suggestions of our work. We also appreciate the reviewer's positive evaluation on our work. Based on the suggestions of the reviewer, we have further added numerical results on second-order and third-order correlation functions of three-body BICs. In addition, the significance of onsite interaction and the boundary effect on the formation of many-body BICs is emphasized. Furthermore, we have incorporated simulation results to assess impacts of losses and lattice length on the observation of three-boson BICs in electric circuits. We believe that the reviewer will find the revised version of our manuscript considerably improved.

1. The correlations of the many-body BICs are lack. Although we know the density distribution of the many-body BICs, we are unclear about the correlations between particles, which are as important as the density distribution. I suggestions the authors give second-order and third-order correlation functions for the three-body BICs.

Reply: We would like to thank the reviewer for the comment. Following the reviewer's suggestion, we calculate second-order $g^{(2)}(i, j) = \frac{\langle a_i^\dagger a_j^\dagger a_i a_j \rangle}{\langle a_i^\dagger a_i \rangle \langle a_j^\dagger a_j \rangle}$ and third-order $g^{(3)}(i, j, k) = \frac{\langle a_i^\dagger a_j^\dagger a_k^\dagger a_i a_j a_k \rangle}{\langle a_i^\dagger a_i \rangle \langle a_j^\dagger a_j \rangle \langle a_k^\dagger a_k \rangle}$ correlation functions for the three-body BICs, as shown in Figs. R(1a) and (1b). Here, $\hat{a}_i$ ($\hat{a}_i^\dagger$) corresponds to the annihilation (creation) operator at the i th site, and $\langle \cdot \rangle$ denotes the expectation value with respect to the eigenstate of three-body BICs. It is clearly shown that the maximum values of second-order and third-order correlation functions appear at $g^{(2)}(1,1)$, $g^{(2)}(L,L)$ and $g^{(3)}(1,1,1)$, $g^{(3)}(L,L,L)$, indicating that three-boson BICs possess a high probability of detecting all bosons at the same boundary site.

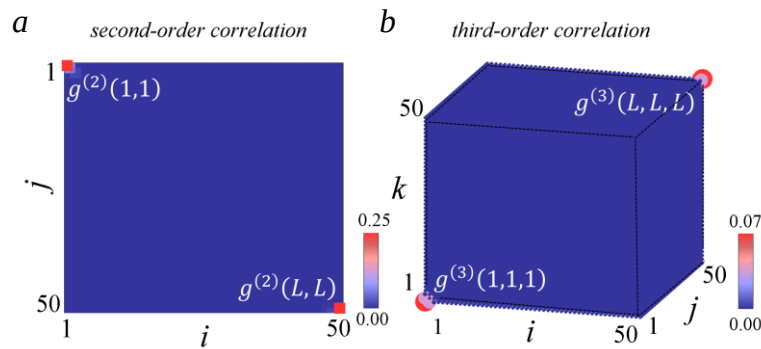


Figure R1. (a) and (b) Numerical results of second-order and third-order correlation functions for three-body BICs.

Action taken:

- We have added numerical results of second-order and third-order correlation functions for the three-body BICs in Figs. (d1) and (d2).

- In the page 6 of the revised manuscript, we have added the following discussion to illustrate the second-order and third-order correlation functions for the three-body BICs: “*Furthermore,*

we also calculate the second-order $g^{(2)}(i, j) = \frac{\langle a_i^\dagger a_j^\dagger a_i a_j \rangle}{\langle a_i^\dagger a_i \rangle \langle a_j^\dagger a_j \rangle}$ and third-order $g^{(3)}(i, j, k) =$

$\frac{\langle a_i^\dagger a_j^\dagger a_k^\dagger a_i a_j a_k \rangle}{\langle a_i^\dagger a_i \rangle \langle a_j^\dagger a_j \rangle \langle a_k^\dagger a_k \rangle}$ correlation functions for the three-body bound state, as shown in Figs. (d1)

and (d2) with $U=1.7$, $t=1$, and $L=50$. Here, $\hat{a}_i$ ($\hat{a}_i^\dagger$) corresponds to the annihilation (creation) operator at the i th site, and $\langle \cdot \rangle$ denotes the expectation value with respect to the eigenstate of three-body BICs. It is clearly shown that the maximum values of second-order and third-order correlation functions appear at $g^{(2)}(1,1)$, $g^{(2)}(L,L)$ and $g^{(3)}(1,1,1)$, $g^{(3)}(L,L,L)$, indicating that three-boson bound states possess a high probability of detecting all bosons at the same boundary site.”.

2. Although the authors tried to give mechanism for many-body BICs in the supplementary information, their descriptions are still rough and cannot well explain the basic mechanism. I believe both the onsite interaction and the boundary effect play important role in the formation of many-body boundary BICs. In the periodic boundary condition, there should be no such many-body BIC at all. The authors emphasize this point in some place of their main text. Furthermore, it would be better for the authors to give clearer explanation of the mechanism.

Reply: We would like to thank the reviewer for the comment. It is true that both the onsite interaction and boundary effect play crucial roles in the formation of many-body BICs, while no such BICs can be found in a lattice model with periodic boundary conditions. In the case of a lattice model with periodic boundary conditions, all lattice sites are equivalent, thus lacking an effective interface for the emergence of many-body localized states. Conversely, in a lattice model with open boundaries, the distinct nature of boundary sites compared to bulk sites results in the particle localization at the boundaries. Consequently, the presence of onsite interactions further facilitates the formation of many-body BICs. The detailed explanation on the mechanism of many-body BICs is provided in the following comment.

Action taken:

- In page 7 of the revised manuscript, we have added the following discussion to emphasize the absence of many-body BICs in the lattice model with periodic boundary conditions:
“Furthermore, it is important to emphasize that both the onsite interaction and boundary effect play crucial roles in the formation of many-body BICs, while no such BICs can be found in a lattice model with periodic boundary conditions. In the case of a lattice model with periodic boundary conditions, all lattice sites are equivalent, thus lacking an effective interface for the emergence of many-body localized states.”.

3. It seems that such many-body BICs exist in systems with as least three particles. Could the authors explain why two-particle BICs are absent?

Reply: We would like to thank the reviewer for the comment. The existence of two-particle BICs requires the boundary effect, particle interaction, and *defect potential* at lattice boundaries [*J. Phys. Condens. Matter* 25, 235601 (2013)]. Without the boundary defect potential, the two-particle BICs

cannot appear. Conversely, we find that the presence of boundary defective potential is unnecessary for the existence of many-body boundary BICs when the number of bosons is three or greater.

To gain further insight into the formation of two-particle BICs, we present the Fock space of two bosons in the 1D lattice model with boundary defective potential and particle interaction, as shown in Fig. R2(a). The orange dots represent the aggregation states of two bosons with interaction U , and the green dots correspond to states with one boson located at boundary site with the defective potential V . Thus, through a combined effect of defect potentials and interactions, it is possible for two bosons to localize perfectly at the corner in Fock space (the black dot with $U+V$) and form BICs. Different from the two-particle system, many-boson eigenstates possess more kinds of particle aggregations. As depicted in Fig. R2(b), we illustrate the Fock space of three bosons, where red dots represent three-boson aggregation states with an interaction strength of $3U$, while orange and white dots correspond to two-boson aggregation states with an interaction strength of U and single-particle scattering states without interactions. By considering the interacting two-boson aggregation states (orange dots) as an effective defect potential for three-boson aggregation states, it becomes feasible to achieve many-body BICs without additional defects.

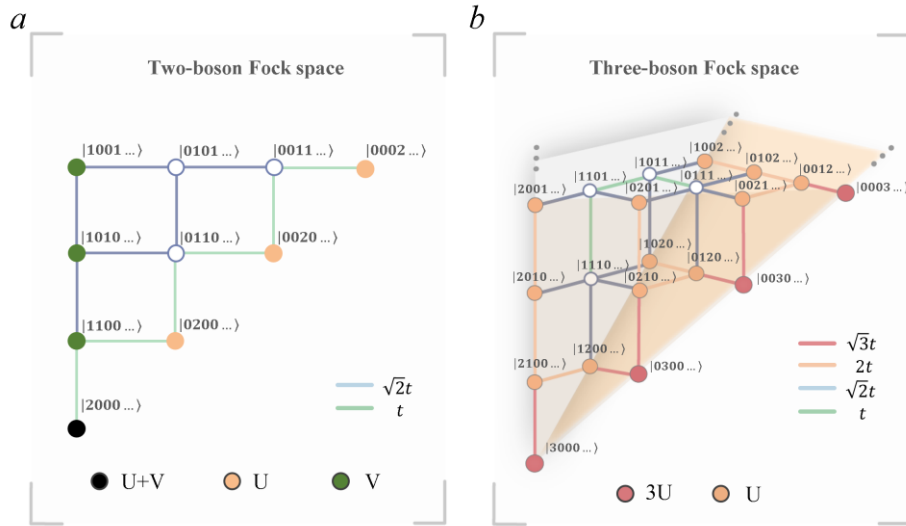


Figure R2. (a) and (b). Two-boson and three-boson Fock spaces.

Action taken:

- In page 7 of the revised manuscript, we have added the following discussion to further illustrate different requirements on the formation of two-particle and many-particle BICs: *“In addition, Ref. 36 has demonstrated that the existence of two-particle boundary BICs requires the*

boundary effect, particle interaction, and defect potential at lattice boundaries. Without the boundary defect potential, the two-particle boundary BICs cannot appear. Conversely, we observe that the presence of boundary defective potential is unnecessary for the existence of many-body boundary BICs when the number of bosons is three or greater. Such a distinction stems from the existence of three types of three-boson states, the single-PSSs, two-BASs and three-BASs, with different interaction energies. In this case, the two-BASs with the interaction energy being U can be regarded as an effective defect potential of three-BASs, making the formation of three-boson BICs do not require any defect potential (See Supplementary Note 4 for details).”.

- In Supplementary Note 4, we have added the detailed discussion on the difference between two-particle and three-particle BICs.

4. In the caption of Figure 1, the authors choose some special states to present their results. Could the authors explain why they choose the states with indices “4302, 22051, 22097 and 21934”?

Reply: We would like to thank the reviewer for the comment. The reason that we choose these states is that we want to illustrate the density distribution of eigenstates for three bosons with different interaction energies. Specifically, the eigen-energy of the state at 4302 equals to $\varepsilon = -2.05$, corresponding to the scenario with three non-interacting bosons locating at different lattice sites. The eigenstates with the indices being 22051 and 22097 at $\varepsilon = -2.05$ and 6.29 represent cases where two and three bosons locate at the same site, resulting in the interaction energy of U and $3U$. We call these three different types of three-boson states as the single-particle scattering states (single-PSSs), two-Boson aggregation states (two-BASs) and three-Boson aggregation states (three-BASs), respectively. In addition, the state at 21934 with $\varepsilon = 5.87$ corresponds to the three-boson BIC. It is worth noting that the density distributions of other single-PSSs, two-BASs, and three-BASs possess the similar characteristics to those shown in Figs. 1(c1-c3).

Action taken:

- In page 4 of the revised manuscript, we have revised this expression to avoid the controversy: *“Numerical results of the particle density distributions for three-boson states at $\varepsilon = -2.05$ (single-PSS), 6.29 (two-BAS), 7.47 (three-BAS), and 5.87 (three-boson BIC), respectively.”.*
- In page 5 of the revised manuscript, we have emphasized that the density distributions of other

single-PSSs, two-BASs, and three-BASs possess similar characteristics to those shown in Figs.

1(c1-c3): *“It is worth noting that the density distributions of other single-PSSs, two-BASs, and three-BASs possess the similar characteristics to those shown in Figs. 1(c1-c3).”*

5. In Eq. (4), there is a strange number “7.6”. In the supplementary information, “7.6” suddenly appear in Eq. (S18), before which there is no such number. Where this number comes from? What parameters are this number related to?

Reply: We would like to thank the reviewer for the comment. Eq. (4) and Eq. (S18) correspond to the circuit eigenequation expressed with respect to the circuit node of $V_{|...0,1,1,1,0...}$. In particular, by applying the Kirchhoff's law on the circuit node of $V_{|...0,1,1,1,0...}$, we have

$$\begin{aligned} \frac{1}{L_g} V_{|...0,1,1,1,0...} = & \omega^2 C [(V_{|...0,1,1,1,0...} - V_{|...1,0,1,1,0...}) + (V_{|...0,1,1,1,0...} - V_{|...0,1,1,0,1...})] \\ & + \omega^2 \sqrt{2} C [(V_{|...0,1,1,1,0...} - V_{|...0,0,2,1,0...}) + (V_{|...0,1,1,1,0...} - V_{|...0,2,0,1,0...}) + (V_{|...0,1,1,1,0...} - V_{|...0,1,0,2,0...})] \\ & + (V_{|...0,1,1,1,0...} - V_{|...0,1,2,0,0...}) + \omega^2 C U_{single} V_{|...0,1,1,1,0...}. \end{aligned} \quad (R7)$$

By simplifying the right-hand side of Eq. (R7), we have

$$\begin{aligned} \frac{1}{L_g} V_{|...0,1,1,1,0...} = & \omega^2 C (2 + 4\sqrt{2}) V_{|...0,1,1,1,0...} - \omega^2 C V_{|...1,0,1,1,0...} - \omega^2 \sqrt{2} C (V_{|...0,0,2,1,0...} + V_{|...0,2,0,1,0...} \\ & + V_{|...0,1,0,2,0...} + V_{|...0,1,2,0,0...}) - \omega^2 C V_{|...0,1,1,0,1...} + \omega^2 C U_{single} V_{|...0,1,1,1,0...}. \end{aligned} \quad (R8)$$

Then, by shifting the term $\omega^2 C (2 + 4\sqrt{2}) V_{|...0,1,1,1,0...}$ (with $\sqrt{2} \approx 1.4$) from right to left and subsequently dividing the equation by the factor $\omega^2 C$, we obtain

$$\begin{aligned} \left(\frac{1}{\omega^2 C L_g} - 7.6 \right) V_{|...0,1,1,1,0...} = & -V_{|...1,0,1,1,0...} - \sqrt{2} C V_{|...0,0,2,1,0...} - \sqrt{2} C V_{|...0,2,0,1,0...} - \sqrt{2} C V_{|...0,1,0,2,0...} \\ & - \sqrt{2} C V_{|...0,1,2,0,0...} - C V_{|...0,1,1,0,1...} + U_{single} V_{|...0,1,1,1,0...}, \end{aligned} \quad (R9)$$

which is in the same form to Eq. (4) and Eq. (R18).

Action taken:

■ In Supplementary Note 6, we have provided the detailed deviation of Eq. (4).

6. How the number of many-body boundary BICs increases with the number of particles?

Reply: We would like to thank the reviewer for the comment. When the number of particles increases, the number of many-body BICs remains invariant (equals to two). This is attributed to the fact the many-body BICs correspond to states that all particles are localized around the endpoints of the 1D lattice. Hence, even for a larger number of particles, only two boundary BICs persist.

Action taken:

- In page 9 of the revised manuscript, we have added the following discussion to emphasize this point: *“In addition, the number of these boundary-localized BICs remains unchanged with the increase in the number of particles, and their eigen-energies are degenerated when the lattice length is large enough.”*.

7. In Figure 3, the experimental results depart quite a lot from the theoretical results. The authors explain that the deviation arises from loss effects in the circuit sample. Could the authors also give numerical results that take the loss effects into account?

Reply: We would like to thank the reviewer for the comment. Following the reviewer’s suggestion, we perform the simulation by taking the loss effects into account (we used the AD633JRZ spice model and the parasitic resistance of inductor is 220 ohms), as shown in Fig. R3. The red, blue and orange dots represent the experimental results, simulation results without loss, and simulation results with loss, respectively. It can be seen that the simulation results considering loss effects are consistent with experimental results.

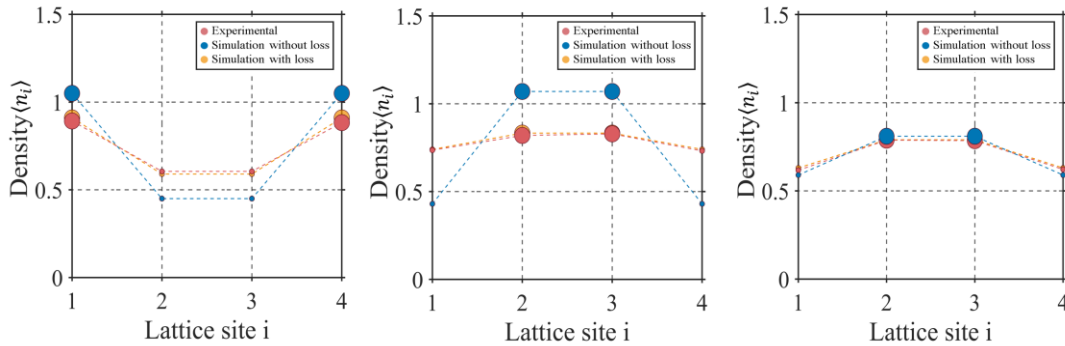


Figure R3. The red, blue and orange dots represent the experimental results, simulation results without loss, and simulation results with loss, respectively.

Action taken:

- In Fig. 3 of the revised manuscript, we have added simulation results of electric circuits with loss effects.
- In page 13 of the revised manuscript, we have added the following discussion to illustrate the influence of loss effects: *“In addition, we perform simulations by taking loss effects into account (we used the AD633JRZ spice model and the parasitic resistance of inductor is 220 ohms), as shown by orange dots in Figs. 3(d1)-3(d3). It can be seen that the simulation results*

considering loss effects are consistent with experimental results.”.

8. The system size is quite small in the experiments. It is hard to claim the state in Figure 3(d1) as BIC with such small systems. Could the authors present experimental results of three particles in six lattices or more?

Reply: We would like to thank the reviewer for the comment. As the lattice length increases, the number of active components such as operational amplifiers and multipliers increases rapidly, leading to enhanced nonlinear instability of the system.

To further elucidate the impact of lattice length on the simulation of three-boson BICs, we conducted simulations considering loss effects for three-boson circuits with varying lattice lengths ($L=4, 6$, and 7), as depicted in Fig. R4. It is evident that three-boson BICs exhibit a similar localization effect near the boundaries of the lattice model. Consequently, it is demonstrated that distributions of effective particle density in systems with $L=4$ can also manifest a comparable spatial profile to those observed in longer lattice lengths. Hence, our experimental findings effectively emulate three-boson BICs.

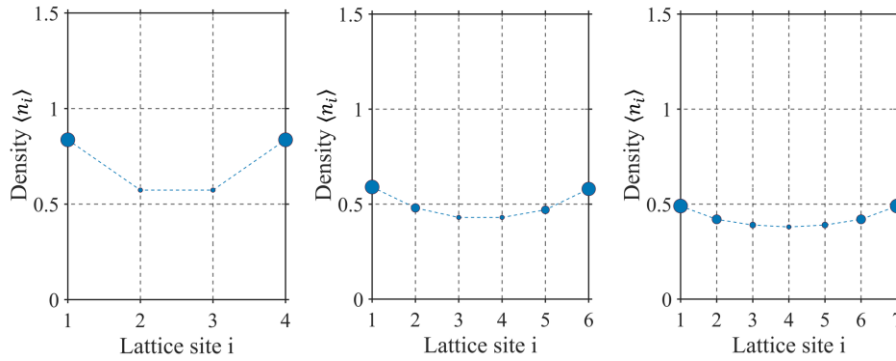


Figure R4. Simulation results of electric circuits with the lattice length being $L=4, 6$, and 7 .

Action taken:

- In Supplementary Note 7, we have added simulation results of three-boson circuits with different lattice lengths
- In page 13 of the revised manuscript, we have added the following discussion to illustrate the influence of lattice length on three-boson BICs: *“Furthermore, as the lattice length increases, the number of active components such as operational amplifiers and multipliers increases rapidly, leading to enhanced nonlinear instability of the system. To further illustrate the influence of lattice length on three-boson BICs, we conducted simulations considering loss*

effects for three-boson circuits with varying lattice lengths (See Supplementary Note 7 for details). It is shown that distributions of effective particle density in circuits with $L=4$ can also manifest a comparable spatial profile to those observed in longer lattice lengths. Hence, our experimental findings effectively emulate three-boson BICs.”.

9. The authors may give a section in the supplementary information to explain how and why they can excite the BICs in experiments since they are lying in the continuum spectrum.

Reply: We would like to thank the reviewer for the comment. We can directly excite the three-boson BIC in circuits without needing to excite it from the continuum spectrum. In particular, three-body BICs refer to the state where all particles are localized at the lattice boundary, corresponding to Fock-space basis vectors of $|3, 0, 0, \dots\rangle$ and $|\dots, 0, 0, 3\rangle$. We note that the Kirchhoff equation of our designed circuit is in the same form to the three-boson eigen-equation in Fock space, where eigen-frequencies of the circuit correspond to eigen-energies of lattice model and the voltage distribution is mapped to the three-boson eigen-profile in Fock space. Therefore, to excite the BICs in circuits, we only need to excite the effective Fock-space basis vector of $|3, 0, 0, \dots\rangle$ or $|\dots, 0, 0, 3\rangle$ by injecting the voltage signal into the circuit node $V_{|3,0,0,\dots\rangle}$ or $V_{|\dots,0,0,3\rangle}$. Then, by changing the input frequency of the voltage, it becomes evident that the voltage signal reaches its maximum amplitude when resonating with the eigenenergy of three-boson BICs, thereby affirming direct excitation of BICs.

Action taken:

- We have added a new section in the Method to explain how and why we can excite the BICs in experiments: *“In addition, it is noted that we can directly excite the three-boson BIC in circuits without needing to excite it from the continuum spectrum. In particular, three-body BICs refer to the states where all particles are localized at the boundary, corresponding to Fock-space basis vector of $|3, 0, 0, \dots\rangle$ and $|\dots, 0, 0, 3\rangle$. We note that the Kirchhoff equation of our designed circuit is in the same form to the three-boson eigen-equation in Fock space, where eigen-frequencies of the circuit correspond to eigen-energies of lattice model and the voltage distribution is mapped to the three-boson eigen-profile in Fock space. Therefore, to excite the BICs in circuits, we only need to excite the effective Fock-space basis vector $|3, 0, 0, \dots\rangle$ or $|\dots, 0, 0, 3\rangle$ by injecting the voltage signal into the circuit node $V_{|3,0,0,\dots\rangle}$ or*

$V_{|...0,0,3\rangle}$. Then, by changing the input frequency of the voltage, it becomes evident that the voltage signal reaches its maximum amplitude when resonating with the eigenenergy of three-boson BICs, thereby affirming direct excitation of BICs.”.

10. There are some typos in the main text and supplementary information, such as “these two five-boson bound states are boundry BICs”, “(a) and (b) present two-boson and three-boson Fock spaces”. The authors should carefully check their manuscripts.

Reply: We would like to thank the reviewer for the comment. We have modified these typos in the main text and supplementary information

Response to comments of the reviewer #3

In the manuscript entitled “Boundary many-body bound states in the continuum”, the authors propose a standard Bose-Hubbard simple model in a homogeneous one-dimensional lattice, where three or more interacting bosons can concentrate on the boundary and the eigenenergy of multi-boson bound states can embed within the continuous energy spectra of extended scattering states. The authors further experimentally simulate the many-body BICs based on a mapping to nonlinear circuit networks. The novelty of this work is that it provides a new mechanism to realize many-body BIC and implements it in nonlinear electric circuits, which enables it to be published in Communications Physics. There are some concerns about the manuscript at this current stage.

Reply: We would like to thank the reviewer for the careful review and valuable suggestions of our work. We also appreciate the reviewer's positive evaluation of our work. Based on the suggestions of the reviewer, we have conducted a further comparative analysis of the formation mechanisms of two-body BICs and many-body BICs. We have also presented the divergence tendency on the imaginary part of eigen-energy of three-boson BIC and elaborated on the interaction threshold for boundary-localized BICs. Furthermore, we have elucidated the methodology for mapping nonlinear circuits onto the quantum lattice model. We believe that the reviewer will find the revised version of our manuscript considerably improved.

1. A theoretical comparison in detail with Refs 35 and 36 will be appreciated for the readers who are not familiar with the many-body BICs. Moreover, the statement “the two-BASs with interaction U (two blue shaded areas) emulate the effect of the defect potential of three-BASs with the interaction strength being $3U$ (red shaded areas)” in Supplementary note 4 is not very clarified (similar statement also appear in the main text).

Reply: We would like to thank the reviewer for the comment. Following the review’s suggestion, we have conducted the comparison between many-body BICs and two-particle BICs described in Refs. 35 and 36. In Ref. 35, it is shown that the formation of two-particle BICs localized in the bulk lattice site must require the particle interaction and *defect potential* at the lattice center. In addition, similar to the model considered in our manuscript, Ref. 36 has demonstrated that the existence of two-particle boundary BICs requires the boundary effect, particle interaction, and *defect potential* at lattice boundaries. Without the boundary defect potential, the two-particle boundary BICs cannot appear. Conversely, we observe that the presence of boundary defective potential is unnecessary for the existence of many-body boundary BICs when the number of bosons is three or greater.

To gain further insight into the formation of two-particle boundary BICs in Ref. 36, we have depicted the Fock space of two bosons in Fig. R5(a). The orange dots represent aggregation states of two bosons with interaction U in Fock-space, and the green dots correspond to states with an edge impurity potential V . Through a combined effect of defect potentials and interactions, it is possible for two bosons to localize perfectly at the corner in Fock space (the black dot with $U+V$) and form BICs. Different from the two-particle system, many-body systems can exhibit various forms of particle aggregation. As shown in Fig. R5(b), we illustrate the Fock space of three bosons using red dots to denote three-boson aggregation states with interaction $3U$, orange and white dots represent two-boson aggregation states with interaction U and single-particle scattering states without interactions, respectively. In this case, by considering that the presence of interacting two-boson aggregation states acts as an effective defect potential for three-boson aggregation states, it becomes feasible to achieve many-body BICs without additional defect potentials.

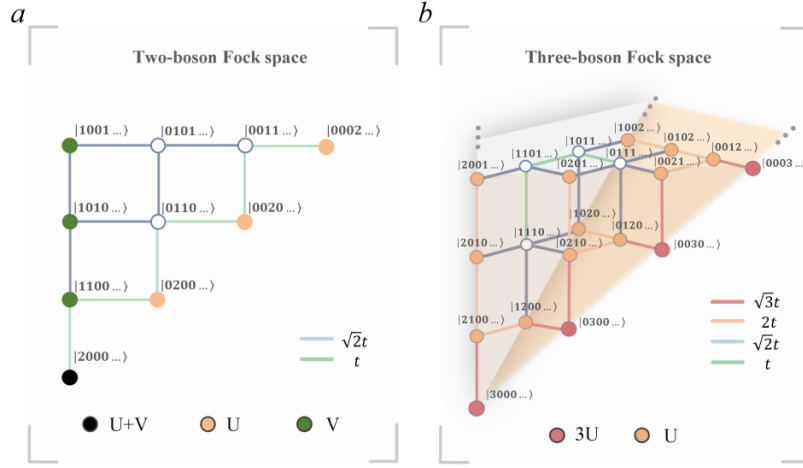


Figure R5. (a) and (b) present two-boson and three-boson Fock spaces.

Action taken:

- In page 8 of the revised manuscript, we have added the following discussion to provide a theoretical comparison in detail with Refs 35 and 36: *“In Ref. 35, it is shown that the formation of two-particle BICs localized at the bulk lattice site requires the coexistence of particle interaction and defect potential at the lattice center. In addition, Ref. 36 has demonstrated that the existence of two-particle boundary BICs requires the boundary effect, particle interaction, and defect potential at lattice boundaries. Without the boundary defect potential, the two-particle boundary BICs cannot appear. Conversely, we observe that the presence of boundary defective potential is unnecessary for the existence of many-body boundary BICs when the number of bosons is three or greater. Such a distinction stems from the existence of three types of three-boson states, the single-PSSs, two-BASs and three-BASs, with different interaction energies. In this case, the two-BASs with the interaction energy being U can be regarded as an effective defect potential of three-BASs, making the formation of three-boson BICs do not require any defect potential (See Supplementary Note 4 for details).”*.
- In Supplementary Note 4, we have added the detailed discussion on the difference between two-particle and three-particle BICs.

2. It may be not proper to write the single-PSS in Eq. (3) as $(\dots 0, 1, 1, 1, 0 \dots)$ since it excludes terms like $(\dots 1, 0, 1, 1, 0 \dots)$. Similar issues for the two-BASs and three-BASs in SI. Furthermore, what is the feature difference for the IPRs of $(\dots 0, 1, 1, 1, 0 \dots)$ and $(\dots 1, 0, 1, 1, 0 \dots)$, as well as of other single-PSS configuration? and for two-BASs and three-BASs?

Reply: We would like to thank the reviewer for the comment. Eigen-equations with respect to all

Fock-space basis vectors constituting the single-PSS exhibit the similar form. For example, the eigenequation with respect to another Fock-space basis vector (...1,0,1,1,0...) is expressed as

$$\begin{aligned} \varepsilon c_{|\dots 0,1,0,1,1,0\dots\rangle} = & -t(c_{|\dots 1,0,0,1,1,0\dots\rangle} + c_{|\dots 0,0,1,1,1,0\dots\rangle} + c_{|\dots 0,1,1,0,1,0\dots\rangle} + \sqrt{2}c_{|\dots 0,1,0,0,2,0\dots\rangle} \\ & + \sqrt{2}c_{|\dots 0,1,0,2,0,0\dots\rangle} + \sqrt{2}c_{|\dots 0,1,0,0,2,0\dots\rangle}), \end{aligned} \quad (\text{R10})$$

which is in the same form with that of (...0,1,1,1,0...) shown in Eq. (3) of the main text. Additionally, the eigen-equations for the Fock-space basis vector (...0,2,0,1,0...) of two-PSSs and (3,0,0...) of three-PSSs can be written as

$$\varepsilon c_{|\dots 0,2,0,1,0\dots\rangle} = -t(\sqrt{2}c_{|\dots 1,1,0,1,0\dots\rangle} + \sqrt{2}c_{|\dots 0,1,1,1,0\dots\rangle} + c_{|\dots 0,2,1,0,0\dots\rangle} + c_{|\dots 0,2,0,1,0\dots\rangle}) + U, \quad (\text{R11})$$

$$\varepsilon c_{|3,0,0\dots\rangle} = -t\sqrt{3}c_{|2,1,0\dots\rangle} + 3U, \quad (\text{R12})$$

which are also in the same form to eigenequations with respect to $c_{|\dots 0,2,1,0\dots\rangle}$ and $c_{|\dots 0,3,0,0\dots\rangle}$ shown in Eqs. (1) and (2) in the Supplementary information. To avoid ambiguity, we have revised the expression in both the main text and SI.

Additionally, *IPR* is used to measure the localization degree of three-boson eigenstates in Fock space, these states are composed of Fock-space basis vectors, such as (...0,1,1,1,0...), (...1,0,1,1,0...) ... for single-PSSs, (...0,2,0,1,0...), (...0,2,1,0...) ... for two-BASs and (...0,3,0...), (3,0,0...) ... for three-BASs. Therefore, the *IPRs* of single-PSSs, two-BASs, and three-BASs depend on the distribution of probability amplitudes on the basis vectors in Fock space. For single-PSSs, the corresponding probability amplitudes are distributed across the basis vectors $|\dots 1_i \dots 1_j \dots 1_k \dots\rangle$ in Fock space. Because the number of the basis vectors $|\dots 1_i \dots 1_j \dots 1_k \dots\rangle$ is larger than that of $|\dots 2_i \dots 1_j \dots 0_k \dots\rangle$ and $|\dots 3_i \dots 0 \dots 0_k \dots\rangle$, the single-PSSs can exhibit a more extended profile in Fock space than that of two-BASs and three-BASs, making the corresponding *IPRs* become relatively small. In contrast, as for two-BASs and three-BASs, the number of the basis vectors $|\dots 2_i \dots 1_j \dots 0_k \dots\rangle$ is greater than that of $|\dots 3_i \dots 0 \dots 0_k \dots\rangle$, indicating that the three-BASs are more localized in Fock space than two-BASs. Thus, *IPRs* of two-BASs are lower than that of three-BASs. Finally, as for three-boson BICs, the probability amplitude of the eigenstate is almost entirely concentrated on $|3, 0, 0, \dots\rangle$ and $|\dots, 0, 0, 3\rangle$, resulting in the highest *IPR* value.

Action taken:

- In Supplementary Note 2, we have added the detailed discussion to explain the difference of *IPRs* for single-PSSs, two-BASs and three-BASs.

- In page 4 of the revised manuscript, we have revised the expression as: “*we can obtain the eigenequation with respect to a Fock-space basis vector $|\dots 0,1,1,1,0 \dots\rangle$ as:*” and “*Eigenequations with respect to the Fock-space basis vectors related to two-PSSs and three-PSSs are presented in Supplementary Note 1.*”.

3. As the authors follows “other numerical method on the demonstration of BICs [40, 41]” to plot Fig. 1(e) with non-Hermitian on-site terms, but there is no ultra deep profile or divergence tendency in Fig. 1(e). Why?

Reply: We would like to thank the reviewer for the comment. The reason that the absence of the divergence tendency is due to the small value of n_l with artificial losses used in Fig. 1(e). As shown in Fig. R6, we calculate the variation of the imaginary part of the eigen-energies for three-boson eigenstates as a function of n_l toward large values. It is shown that the divergence tendency appears for the three-boson BICs (the blue dots).

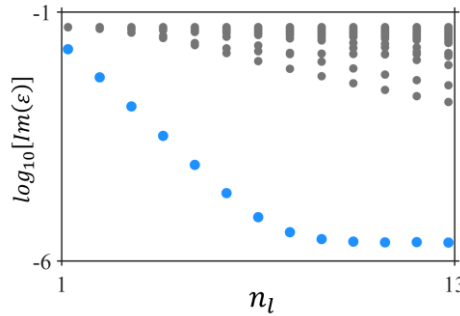


Figure R6. Numerical results on the variation of imaginary parts of eigen-energies as a function of the size for the lossless region with $U=1.7$, $t=1$, and $L=50$.

Action taken:

- We have replotted Fig. 1(f) in the main text to illustrate the divergence tendency of the imaginary part of three-boson eigen-energies as a function of n_l toward large values.
- In page 7 of the revised manuscript, we have added the following discussion to illustrate the divergence tendency of the imaginary part of three-boson eigen-energies: “*It is shown that the divergence tendency appears for the three-boson BICs (the blue dots) with n_l reaching to 11.*”.

4. How to determine the threshold of $U=1.92$? It seems to be very hard to distinguish between BIC and simple boundary state, both of which share the same physical origin. In this regard, why are these specific three-BASs the BICs?

Reply: We would like to thank the reviewer for the comment. The distinguish between boundary BICs and other boundary states is that BICs possess the perfect boundary localization, where the particle density is exponentially decreased to zero away from boundary. In addition, the perfectly localized boundary states can also become bound states *out of* the continuum (BOCs), where the corresponding eigen-energies are outside the three-boson continuum spectrum. *It is noted that the interaction strength U can determine whether the eigen-energy of perfectly localized boundary states can embed within the continuum spectrum.* Specifically, as the interaction strength increases, the eigen-energy of boundary bound states can go outside the continuum spectrum.

In order to determine the interaction threshold of this bound state within the continuum band, we conducted a parameter scan of U with a lattice length of $L=100$. Figs. R7a-R7c present the calculated eigen-spectra of the three-boson model around the eigen-energy of boundary bound states with the interaction being $U=1.91$, 1.92 , and 1.93 , respectively. It is clearly shown that as the interaction increases, the energy of boundary bound states (red dots) gradually move out of the continuum spectrum. At $U=1.92$, the boundary state is at the edge of the continuum spectrum, being the interaction threshold of three-boson BICs.

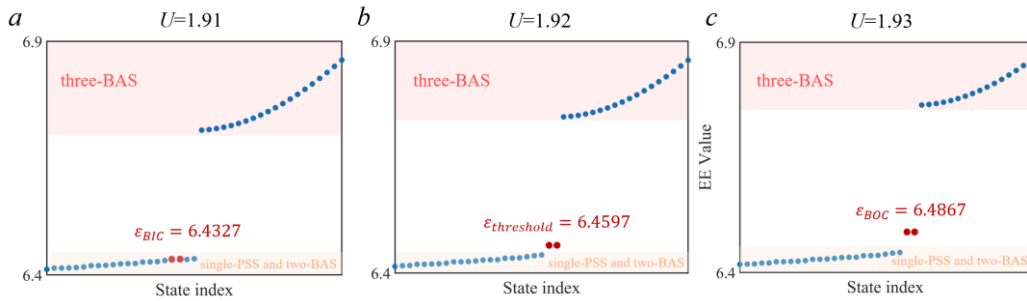


Figure R7. The local energy spectrum of the three-boson near the eigen-energy of boundary-localized states with interactions of 1.91, 1.92, and 1.93 for $L=100$.

In addition, these two specific three-BASs (marked by red dots in Fig. R7) are identified as BICs because their eigenenergies are embedded within the continuum spectrum, while these two three-BASs with perfect boundary localization do not couple with other extended eigenmodes in the continuum band.

Action taken:

- In Supplementary Note 3, we have added the detailed discussion to explain how to determine the interaction threshold of three-body BICs.
- In Page 7 of the revised manuscript, we have added the following discussion to illustrate the

interaction threshold of three-body BICs: “*Moreover, the discrete eigen-energies of boundary-localized three-boson bound states tend to move outside the continuous eigenspectra as the interaction strength increases, giving a threshold of $U=1.92$ for the existence of three-boson BICs (See details in Supplementary Note 3).*”.

5. When comparing the results of three, four, and five-boson cases, is there a threshold of particle number to observe the boundary BIC under the same U ?

Reply: We would like to thank the reviewer for the comment. At the same interaction strength U , there exists a particle number threshold that ensures the existence of boundary BICs. For example, at $U=1.1$, boundary BICs (BOCs) can be observed for $N \leq 4$ ($N > 4$). In the main text Fig. 2e, we provide the interaction thresholds for BICs under different particle numbers. For the three, four, and five-boson cases, the approximate interaction thresholds we calculated are 1.92, 1.34, and 1.01, respectively. This means that at the same interaction strength $U=1.1$, we can observe the eigenenergy of the boundary BIC for three and four bosons within the continuum band (BIC), whereas for five bosons, the eigenenergy of the boundary bound state go outside the continuum band.

Action taken:

- In page 9 of the revised manuscript, we have added the following discussion to emphasize that there exists a particle number threshold: “*In this case, there also exists a particle number threshold that ensures the existence of boundary BICs at the same interaction strength.*”.

6. The mapping of the nonlinear electric circuits to the one-dimensional lattice model is very difficult to grasp or understand. The lattice model is clear, but how does the mapping work? Moreover, the role of nonlinearity seems important and needs further clarification.

Reply: We would like to thank the reviewer for the comment. Due to the direct mapping of a one-dimensional many-body lattice model to electric circuit is very challenging, we can simulate the Fock-space lattice of few-boson model by nonlinear circuit networks. Specifically, we can design the nonlinear circuit with the corresponding eigen-equation being consistent to that of few-body lattice model in Fock space. Using the Fock-space basis vectors $|\dots 0,1,1,1,0 \dots\rangle$, $|\dots 0,2,1,0 \dots\rangle$, and $|\dots 0,3,0,0 \dots\rangle$ as examples, the corresponding eigen-equations can be expressed as:

$$\varepsilon c_{|\dots,0,1,1,1,0\dots\rangle} = -t(c_{|\dots,1,0,1,1,0\dots\rangle} + \sqrt{2}c_{|\dots,0,0,2,1,0\dots\rangle} + \sqrt{2}c_{|\dots,0,2,0,1,0\dots\rangle} + \sqrt{2}c_{|\dots,0,1,0,2,0\dots\rangle} + \sqrt{2}c_{|\dots,0,1,2,0,0\dots\rangle} + c_{|\dots,0,1,1,0,1\dots\rangle}), \quad (\text{R13})$$

$$\varepsilon c_{|\dots,0,2,1,0\dots\rangle} = -t(\sqrt{2}c_{|\dots,1,1,1,0\dots\rangle} + 2c_{|\dots,0,1,2,0\dots\rangle} + c_{|\dots,0,2,0,1\dots\rangle} + \sqrt{3}c_{|\dots,0,3,0,0\dots\rangle}) + U, \quad (\text{R14})$$

$$\varepsilon c_{|\dots,0,3,0,0\dots\rangle} = -t(\sqrt{3}c_{|\dots,1,2,0,0\dots\rangle} + \sqrt{3}c_{|\dots,0,2,1,0\dots\rangle}) + 3U. \quad (\text{R15})$$

As for the designed nonlinear circuit, circuit eigenequations with respect to three circuit nodes with $V_{|\dots,0,1,1,1,0\dots\rangle}$, $V_{|\dots,0,2,1,0\dots\rangle}$ and $V_{|\dots,0,3,0,0\dots\rangle}$ can be expressed as:

$$(f_0^2/f^2 - 7.6)V_{|\dots,0,1,1,1,0\dots\rangle} = -(V_{|\dots,1,0,1,1,0\dots\rangle} + \sqrt{2}V_{|\dots,0,0,2,1,0\dots\rangle} + \sqrt{2}V_{|\dots,0,2,0,1,0\dots\rangle} + \sqrt{2}V_{|\dots,0,1,0,2,0\dots\rangle} + \sqrt{2}V_{|\dots,0,1,2,0,0\dots\rangle} + V_{|\dots,0,1,1,0,1\dots\rangle}) + U_{single}V_{|\dots,0,1,1,1,0\dots\rangle}, \quad (\text{R16})$$

$$(f_0^2/f^2 - 7.6)V_{|\dots,0,2,1,0\dots\rangle} = -(\sqrt{2}V_{|\dots,1,1,1,0\dots\rangle} + 2V_{|\dots,0,1,2,0\dots\rangle} + V_{|\dots,0,2,0,1\dots\rangle} + \sqrt{3}V_{|\dots,0,3,0,0\dots\rangle}) + U_{two}V_{|\dots,0,2,1,0\dots\rangle}, \quad (\text{R17})$$

$$(f_0^2/f^2 - 7.6)V_{|\dots,0,3,0,0\dots\rangle} = -(\sqrt{3}V_{|\dots,1,2,0,0\dots\rangle} + \sqrt{3}V_{|\dots,0,2,1,0\dots\rangle}) + U_{three}V_{|\dots,0,3,0,0\dots\rangle}, \quad (\text{R18})$$

From Eqs. (R13)-(R15) and Eqs. (R16)-(R18), it can be seen that each node within the circuit corresponds to basis vectors within the Fock space, where the voltage $V_{|n_1, n_2, \dots, n_L\rangle}$ at the circuit node labeled by $|n_1, n_2, \dots, n_L\rangle$ is mapped to the probability amplitude of $c_{|n_1, n_2, \dots, n_L\rangle}$. The U_{single} , U_{two} and U_{three} are proportional to the amplification factors of multipliers used to simulate effective interaction energies of single-particle scattering states, two-boson and three-boson aggregative states. In addition, the eigen-energy (ε) of the three-boson lattice model is directly mapped to eigen-frequency (f) of the nonlinear circuit with $\varepsilon = f_0^2/f^2 - 7.6$. The detailed deviation of Eqs. (R16)-(R18) is provided in [Supplementary Note 6](#), elucidating how the relationship $\varepsilon = f_0^2/f^2 - 7.6$ can be derived. It is noted that the eigenequations with respect to other basis vectors in Fock space also possess the same form to that of other corresponding circuit nodes.

In addition, nonlinear self-feedback circuits are utilized to implement the interaction terms U_{single} , U_{two} and U_{three} of single-PSSs, two-BASs, and three-BASs in the eigen-equations of the circuit by adjusting the amplification factors of the operational amplifier modules and multipliers. Specifically, we can control the amplification factor of the multiplier (through R_1 and R_2 shown in [Fig. S4](#) in the Supplementary Materials) to adjust the effective interaction strength. This allows us to experimentally simulate few-boson BICs with different effective interaction strengths.

Action taken:

- In page 9 of the revised manuscript, we have added the following discussion to explain the mapping of the nonlinear electric circuits to the one-dimensional lattice model: “*Since directly*

mapping a nonlinear circuit to a one-dimensional lattice model is challenging, we first map the one-dimensional lattice model to the Fock space. Then, based on the eigen-equations of the basis vectors in the Fock space, we design the circuit's eigen-equations more directly.”.

- In page 10 of the revised manuscript, we have added the following discussion to clarify the role of nonlinear feedback circuit: “*Specifically, we can control the amplification factor of the multiplier (through R_1 and R_2 shown in Fig. 3(a)) to adjust the effective interaction strength. This allows us to experimentally simulate few-boson BICs with different effective interaction strengths.*”.
- In Supplementary Note 6, we have provided the detailed deviation to illustrate the equivalence of eigen-equations for nonlinear circuit networks and three-body lattice model.

Some typos:

(1) In Line 34, it should be [2, 3]. From Line 116 to 121, it should be Fig. 1 not just Fig.

(2) The caption in Supplementary Figure 5 should be for three-boson states.

Reply: We would like to thank the reviewer for the comment. We have modified these typos in the main text and supplementary information

REVIEWERS' COMMENTS:

Reviewer #1 (Remarks to the Author):

The paper has been revised according to the comment. It can be advised to be published.

Reviewer #2 (Remarks to the Author):

The authors have satisfactorily address my comments and suggestions. I believe this manuscript can be an important work in this field and motivate more works about many-body BICs. I suggest acceptance of this manuscript.

Reviewer #3 (Remarks to the Author):

All my concerns have been perfectly addressed, and thereby I will recommend the current manuscript for publication.

Response Letter to Reviewers

Response to comments of the reviewer #1

The paper has been revised according to the comment. It can be advised to be published.

Reply: We would like to thank the reviewer for the recommendation.

Response to comments of the reviewer #2

The authors have satisfactorily address my comments and suggestions. I believe this manuscript can be an important work in this field and motivate more works about many-body BICs. I suggest acceptance of this manuscript.

Reply: We would like to thank the reviewer for the recommendation.

Response to comments of the reviewer #3

All my concerns have been perfectly addressed, and thereby I will recommend the current manuscript for publication.

Reply: We would like to thank the reviewer for the recommendation.