# Supplementary Information for "Symmetry-Governed Dynamics of Magnetic Skyrmions Under Field Pulses"

**Supplementary Note 1 | Skyrmions stability with demagnetizing field presence**. The skyrmion spin textures discussed in the main text are stabilized using the Hamiltonian in Eq. (1), which does not account for the presence of demagnetizing fields. However, it is well-known[1,2] that in 2D magnetic systems with simple spin textures, the effect of the demagnetizing field can be effectively captured by introducing an easy-plane anisotropy.

To compensate for the influence of the demagnetizing field, an additional easy-axis anisotropy term can be included. In this case, skyrmions are expected to remain stable under the same external magnetic field conditions.

To further investigate the stability of the most representative skyrmions, we explicitly include the demagnetizing field, $\mathbf{B}_\mathrm{d}$, in the Hamiltonian and consider the following potential energy term:

$$\mathcal{H}_\mathrm{pot} = -\int \left[ M_\mathrm{s}(\mathbf{B}_\mathrm{ext} + \mathbf{B}_\mathrm{d}) \cdot \mathbf{n} + \frac{1}{2}\mu_0 M_\mathrm{s}^2 n_\mathrm{z}^2 \right] \mathrm{d}S, \quad (1)$$

where is $\frac{1}{2}\mu_0 M_\mathrm{s}^2$ represents the effective easy-axis anisotropy. Supplementary Fig. 1 shows several types of skyrmions that are stabilized both with and without the inclusion of demagnetizing fields. As the figure illustrates, the values of the external magnetic field required for stability differ only slightly between the two cases. Thus, the demagnetizing fields do not prevent skyrmion stability in 2D magnetic systems.

It's important to emphasize that skyrmion stability in realistic systems (with demagnetizing fields) doesn't require easy-axis anisotropy. Most solutions in the main text remain stable in the presence of demagnetizing fields, even without magnetocrystalline anisotropy, although their stability may shift toward stronger external fields.

More generally, the phenomenon described is independent of the underlying Hamiltonian. Linear motion, rotation, and breathing of magnetic solitons occur in all 2D magnetic systems that support soliton stability. We used the chiral magnet model because it offers a wide range of solitons with diverse topological charges and symmetries, allowing us to study all 2D solitons within a single framework.

**Supplementary Note 2 | Derivation of velocity equations**. Employing the spherical angles $(\Theta, \Phi)$, the magnetization vector of unit length can be parametrized $\mathbf{n} = (\sin\Theta\cos\Phi, \sin\Theta\sin\Phi, \cos\Theta)$. In this case, the Landau-Lifshitz equation[3] takes the following form,

$$\partial_t \Phi \sin\Theta = -\frac{\gamma}{M_\mathrm{s}}\frac{\delta\mathcal{E}}{\delta\Theta},\ \partial_t \Theta \sin\Theta = \frac{\gamma}{M_\mathrm{s}}\frac{\delta\mathcal{E}}{\delta\Phi}, \quad (2)$$

where $\gamma$ is the gyromagnetic ratio. At $t = 0$ the system is in equilibrium, and the magnetization is given by $\Phi_0(\mathbf{r}) = \Phi(\mathbf{r}, 0)$, $\Theta_0(\mathbf{r}) = \Theta(\mathbf{r}, 0)$ satisfying the corresponding Euler-Lagrange equations. Assuming that the amplitude of the pulse is small, $\delta B(t) \ll B_0$, and neglecting the solitons' shape deformation, we obtain the following equations from (2):

$$\partial_t \Phi \sin\Theta_0 = \gamma\delta B \sin\Theta_0,\ \partial_t \Theta \sin\Theta_0 = 0. \quad (3)$$

In the following, we consider two types of dynamics: translational motion and rotational motion of skyrmions. **Translational motion.** For skyrmions moving with a constant velocity $\boldsymbol{v}$, meaning $\Phi(\mathbf{r}, t) = \Phi(\mathbf{r} - \boldsymbol{v}t)$ and $\Theta(\mathbf{r}, t) = \Theta(\mathbf{r} - \boldsymbol{v}t)$, we get from (3) the following equation:

$$-\boldsymbol{v} \cdot \nabla\Phi_0 = \gamma\delta B. \quad (4)$$

Multiplying (4) by magnon density $\mu = (4\pi)^{-1}(1 - \cos\Theta_0)$ and performing the integration we obtain

$$\boldsymbol{v} \cdot \boldsymbol{p} = -\gamma m \delta B, \quad (5)$$

where $\boldsymbol{p}$ and $m$ are momentum and magnonic number defined as follows:

$$\boldsymbol{p} = \int \mu\nabla\Phi_0 \mathrm{d}S,\ m = \int \mu \mathrm{d}S. \quad (6)$$

As shown in Ref.[4], for a translation-invariant Hamiltonian, $\boldsymbol{p}$ and $m$ remain conserved quantities, even when the profiles of $\Theta$ and $\Phi$ deviate from the static configuration.

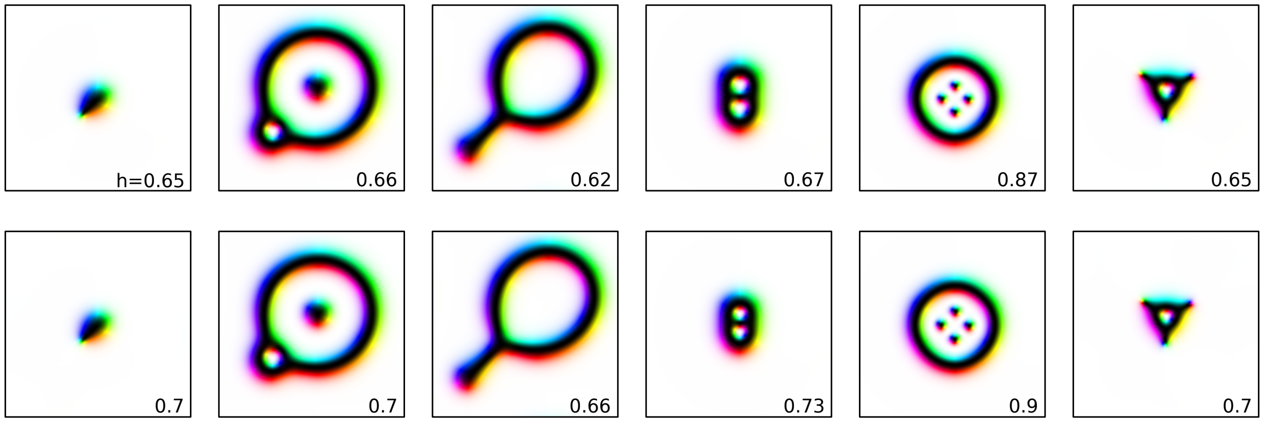
The magnonic number $m$ is invariant under rotations and translations of the texture, but this is not true for the momentum. Under rotation of the skyrmion about $z$-axis, $\boldsymbol{p}$ rotates. Note, momentum $\boldsymbol{p}$ is translation-invariant only if $Q = 0$. Thus, $\mathbf{v} = 0$ for any soliton with $Q \neq 0$. Equation (5) has non-trivial solutions only if $Q = 0$ and $\boldsymbol{p} \neq 0$. The latter implies that $k_\mathrm{s} = 1$. Therefore, all skyrmions that exhibit translational motion under an external field pulse should be topologically trivial and should have low symmetry.

We can write the velocity vector in terms of the speed $v = |\boldsymbol{v}|$ and deflection angle $\psi$: $(v_\mathrm{x}, v_\mathrm{y}) = v(\cos\psi, \sin\psi)$. To determine $v$ and $\psi$, we have to consider (5) and the second equation in (3). Substituting in the latter the rigid translation ansatz gives:

$$-\boldsymbol{v} \cdot \nabla\Theta_0 \sin\Theta_0 = 0. \quad (7)$$

Then we can multiply (7) by $|\mathbf{r} - \mathbf{r}_0|^2$, where $\mathbf{r}_0$ is the skyrmion guiding center and integrate by parts over $xy$-plane:

$$\boldsymbol{v} \cdot \int (\mathbf{r} - \mathbf{r}_0)\mu \mathrm{d}S = v_\mathrm{x} m_\mathrm{y} - v_\mathrm{y} m_\mathrm{x} = 0, \quad (8)$$



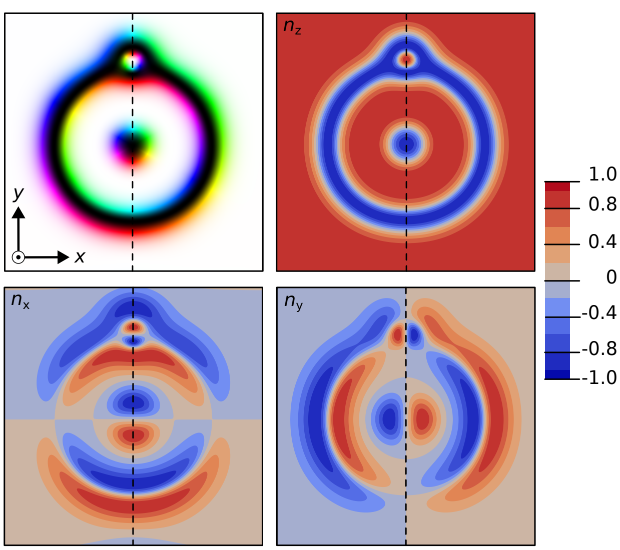

Supplementary Fig. 1. **Skyrmion stability**. Skyrmions of various types are stabilized without (upper row) and with (lower row) demagnetizing fields presence. The dimensionless magnetic field is provided for all spin textures. Simulations were performed for FeGe thin film of size 280nm $\times$ 280nm $\times$ 1nm with periodic boundary conditions in $x$ and $y$ directions.

where $m_{\mathrm{x}}$, $m_{\mathrm{y}}$ are first moments of the magnon density (6). The corresponding vector $\boldsymbol{m} = (m_{\mathrm{x}}, m_{\mathrm{y}})$ transforms in the same way as $\boldsymbol{p}$ under rotation, and are invariant under translation. Thus, we can use (8) to find the deflection angle, $\tan\psi = m_{\mathrm{y}}/m_{\mathrm{x}}$, up to a $\psi \mapsto \psi + \pi$ ambiguity. This means we know the line of motion but not the direction of the soliton along it.

Then, the speed of the skyrmion follows from (5):

$$v = \frac{-\gamma m \delta B}{p_{\mathrm{x}} \cos\psi + p_{\mathrm{y}} \sin\psi}. \tag{9}$$

The ambiguity in $\psi$ can equally be seen as an ambiguity of sign in $v$. Note that as we change the sign of $\delta B$, we flip the direction of motion, but we cannot predict which sign of $\delta B$ will correspond to which direction.




Supplementary Fig. 2. **Skyrmion mirror symmetry**. Skyrmion bag with $Q = 0$, $k_{\mathrm{s}} = 1$ and its magnetization components ($n_{\mathrm{x}}$, $n_{\mathrm{y}}$, $n_{\mathrm{z}}$) are shown. The vertical dashed line corresponds to the skyrmion symmetry axis.

Further simplification of (9) can be made for solitons having an additional mirror symmetry. To exemplify this symmetry, we consider the Snowman skyrmion shown in Supplementary Fig. 2. As one can notice, magnetization components are characterized by additional reflection symmetry: $n_{\mathrm{x}}(x, y) = n_{\mathrm{x}}(-x, y)$, $n_{\mathrm{y}}(x, y) = -n_{\mathrm{y}}(-x, y)$, and $n_{\mathrm{z}}(x, y) = n_{\mathrm{z}}(-x, y)$. The symmetry axis, which coincides with the $y$-axis, so $\psi = 0$. Thus, in the example pictured the direction of motion is along the $x$-axis. For symmetry axis oriented at an arbitrary angle $\psi_0$ to the $x$-axis, $\boldsymbol{m}$ and $\boldsymbol{p}$ are both perpendicular to the line of symmetry, meaning $\psi = \psi_0 \pm \frac{\pi}{2}$. It turns out that this mirror symmetry is widespread among skyrmions. When it is not true, the general formula (9) has to be used.

Interestingly, in the case of translational dynamics of mirror-symmetric skyrmions, we do not have $\boldsymbol{p} \propto \boldsymbol{v}$ like in Newtonian mechanics, but instead an inverse relation. This indicates a significant difference between momentum in Newtonian mechanics and in the theory of magnetic solitons. Nevertheless, $\boldsymbol{v}$ is parallel to $\boldsymbol{p}$, in accordance with numerical observations, as we demonstrate below.

**Rotating motion.** In the case of skyrmion rotation with angular velocity $\omega$, the magnetic texture motion is described as $\Phi(x, y, t) = \Phi(x', y') - \omega t$ and $\Theta(x, y, t) = \Theta(x', y')$ with rotated coordinates:

$$x' = x\cos\omega t - y \sin\omega t, \;\; y' = y\cos\omega t + x\sin\omega t. \tag{10}$$

where $\omega$ is the angular velocity of skyrmion. So the derivative with respect to time can be found as

$$\frac{\partial\Phi}{\partial t} = -\omega - \omega\left(y\frac{\partial\Phi}{\partial x} - x\frac{\partial\Phi}{\partial y}\right). \tag{11}$$

Then substituting this into (3), we get the following equation

$$-\omega - \omega\left(y\frac{\partial\Phi_0}{\partial x} - x\frac{\partial\Phi_0}{\partial y}\right) = \gamma\delta B. \tag{12}$$

Multiplying (12) by $\mu$ and performing integration lead to

$$\omega m + \omega \int \left( y \frac{\partial \Phi_0}{\partial x} - x \frac{\partial \Phi_0}{\partial y} \right) \mu \mathrm{d}S = -\gamma m \delta B. \quad (13)$$

The integral in (13) is nothing but angular momentum, $l$, employing which, the solution for the angular velocity can be found from (13):

$$\omega = \frac{\gamma m \delta B}{l - m}. \quad (14)$$

From (18) follows that $l$ takes non-zero values for skyrmions of arbitrary $k_{\mathrm{s}}$. However, the Thiele equation has nontrivial solutions, $\omega \neq 0$, only for skyrmions with $k_{\mathrm{s}} > 1$.

To use the formula (9), (14), one must compute the corresponding momentum integrals. However, this might represent a challenge due to the non-differentiability of $\Phi$ at skyrmion cores, $\Theta = 0$ and $\Theta = \pi$.

**Supplementary Note 3 | Calculation of momentum integrals**. The momentum $\boldsymbol{p} = (p_{\mathrm{x}}, p_{\mathrm{y}})$ (6) can be rewritten in terms of the topological density:

$$\begin{aligned} p_{\mathrm{x}} &= -\int q y \mathrm{d}S + \int \mu y \varepsilon_{ij} \partial_i \partial_j \Phi_0 \mathrm{d}S, \\ p_{\mathrm{y}} &= \int q x \mathrm{d}S - \int \mu x \varepsilon_{ij} \partial_i \partial_j \Phi_0 \mathrm{d}S, \end{aligned} \quad (15)$$

where $\varepsilon_{\mathrm{xy}} = -\varepsilon_{\mathrm{yx}} = 1$ and $\varepsilon_{\mathrm{xx}} = \varepsilon_{\mathrm{yy}} = 0$.

The method for computing the last integrals in (15) depends on the type of cores the particular skyrmion has. An important characteristic here is the winding number:

$$\nu = \lim_{\epsilon \to 0} \frac{1}{2\pi} \oint_{C_\epsilon} \nabla \Phi \cdot \mathrm{d}\mathbf{l}, \quad (16)$$

where the integration contour $C_\epsilon$ is defined by $\Theta(\mathbf{r}) = \pi - \epsilon$.

Some skyrmions can have a few cores, as shown in Supplementary Fig. 2 **b**, **c** in the main text. While axially symmetric $\pi$-skyrmion **a** has only one core, chiral droplet **b** and antiskyrmion **c** have two and three cores, respectively. These multicore situations are quite generic, and more complex solitons, as a rule, are characterized by more cores. In some cases, a set of points with $\Theta(\mathbf{r}) = \pi$ can represent a curve, as for skyrmionium and other skyrmion bags. In the former and the latter cases, we will refer to such cores as simple cores and complex cores, respectively.

In the case when skyrmions have simple cores only, the last integrals in (15) are proportional to the delta function taken at the core of solitons $\Theta = \pi$ and multiplied by the corresponding vector field vorticity, $\nu_i$, around the core:

$$p_{\mathrm{x}} = -\int q y \mathrm{d}S + \sum_i \nu_i y_i, \;\; p_{\mathrm{y}} = \int q x \mathrm{d}S - \sum_i \nu_i x_i, \quad (17)$$

where $\mathbf{r}_i = (x_i, y_i)$ is the $i$-th core of the skyrmion. It is from this expression that we can see that under a translation $\mathbf{r} \mapsto \mathbf{r} + (a_{\mathrm{x}}, a_{\mathrm{y}})$, the contributions from the integral and the sum are equal and one has $\boldsymbol{p} \mapsto \boldsymbol{p} + 8\pi Q(-a_y, a_x)$, and thus $\boldsymbol{p}$ is only translation-invariant if $Q = 0$. This statement was made more generally in Ref.[4], but without taking into account the delta-function contributions. Similarly, the angular momentum, $l$, can be calculated employing the topological charge density as

$$l = -\frac{1}{2} \int q r^2 \mathrm{d}S + \frac{1}{2} \sum_i \nu_i r_i^2. \quad (18)$$

Complex cores can be split into two cases. In the first case, when the set of points with $\Theta(\mathbf{r}) = \pi$ forms a single closed curve, e.g. skyrmionium, we can always define $\Theta > \pi$ on one side $\Theta < \pi$ on the other so that $\Phi$ has no singularities, and there is thus no contribution to the integrals above. We can say these cores have no winding. After removing such cores, we have the second case, where $\Theta = \pi$ forms a curve with ends and $\Phi$ may be discontinuous across the curve. In this case the winding can be defined as in (16) and it will in general be non-zero.

This situation is exemplified by Supplementary Fig. 2, where there are three curves where $\Theta = \pi$ forming a 'figure-eight' shape. By noting that $\Phi$ increases anticlockwise along each of these curves, we see that the outer curve can be seen as a zero-winding core, and the central bridge separating the two closed regions is an extended core with non-zero winding.

While the formula for the contribution of these cores to the winding is more complicated, it has the same behavior under translation as simple cores. This means that again $\boldsymbol{v} = 0$ for any $Q \neq 0$ texture.


**Supplementary References**

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