Supplementary Information for: Noise-augmented Chaotic Ising Machines for Combinatorial Optimization and Sampling

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I. Supplementary Note 1

In this Supplementary Note, we describe the details of the Kibble-Zurek Mechanism (KZM) results shown in FIG. 4. Although we closely follow the arguments presented in Ref. [1], our purpose is to provide precise details about our own measurement procedure.

According to the scaling hypothesis in phase transitions, macroscopic quantities such as relaxation times and correlation lengths diverge at the critical point. These quantities typically exhibit a power-law relationship with respect to a tuning parameter, λ. In our context, λ is the temperature, T, or its inverse, β. The power-law dependence of key quantities with respect to λ around the critical point can usually be justified by notions of scale-invariance and self-similarity:

$$\xi(\lambda) = \frac{\xi_0}{|\lambda_c - \lambda|^{\nu}} \tag{S1}$$

where ξ is the correlation length, ν is the critical exponent for correlation lengths, and

$$\tau(\lambda) = \frac{\tau_0}{|\lambda_c - \lambda|^{z\nu}} \tag{S2}$$

where τ is the relaxation time and $z\nu$ is the critical exponent for the relaxation time. Consider now a general annealing schedule, $\lambda(t)$ starting from an initial, $\lambda(t = 0) = \lambda_i$

and ending at the critical $\lambda(t = t_a) = \lambda_c$. Since the key quantities of interest diverge near the critical point, the system remains roughly in equilibrium until it approaches the critical point. The non-equilibrium dynamics of interest occur close to the critical point, thus any annealing schedule $\lambda(t)$ can be linearized around λ_c, which is reached at $t = t_a$:

$$\lambda(t) = \lambda_c + \alpha(t - t_a) + O[(t - t_a)^2] \tag{S3}$$

where α is a key quantity representing the velocity of the anneal as the system approaches the critical point:

$$\alpha = \left.\frac{d\lambda(t)}{dt}\right|_{t=t_a} \tag{S4}$$

As the system is annealed towards the critical point, the equilibration time increases. If the initial λ is far away from the critical point, the system quickly relaxes to its equilibrium and can follow the annealing schedule in quasi-equilibrium or adiabatically. There comes a point, $\hat{t}$, when the remaining annealing time, $t_a - \hat{t}$ is precisely equal to the relaxation time of the system, $\tau[\lambda(\hat{t})]$. At this point, any further change in λ that increases the relaxation time results in the system not being able to equilibrate by the end of the annealing. The idea of KZM, then, is to find this "freeze-out" boundary $\hat{t}$ and to relate non-equilibrium correlation lengths to equilibrium correlations lengths via $\xi[\lambda(\hat{t})]$, which is then related to a density of topological defects via dimensional analysis. Based on this sketch, we proceed to find $\hat{t}$ by equating $|t_a - \hat{t}|$ to $\tau(\hat{\lambda})$ using Eq. S2 and Eq. S3. We obtain:

$$\hat{t'} = \left(\tau_0 \alpha^{-z\nu}\right)^{\frac{1}{1+z\nu}} \tag{S5}$$

where $\hat{t'} = t_a - \hat{t}$. We then find $\lambda(\hat{t'})$ as:

$$\lambda(\hat{t'}) = \lambda_c(1 - \tau_0' \alpha^{\frac{1}{1+z\nu}}) \tag{S6}$$

where τ_0' is a modified τ_0 with unimportant constants that do not affect the exponents. We then obtain the correlation lengths at freeze-out from:

$$\xi[\lambda(\hat{t})] = \epsilon_0' \, \alpha^{\frac{-\nu}{1+z\nu}} \tag{S7}$$

where ϵ_0' is a modified ϵ_0 with unimportant constants. Finally, ξ is related to the density of topological defects through dimensional analysis of defect dimension d and system dimension D:

$$\frac{\xi^d}{\xi^D} = \frac{1}{\epsilon_0'^{(D-d)}} \, \alpha^{\left(\frac{\nu}{1+z\nu}\right)(D-d)} \tag{S8}$$

Eq. S8 predicts a power-law relationship for the number of "defects" as a function of quench "velocity", α in terms of equilibrium critical exponents. Defects in the context of Ising models are then relatable to measurable quantities such as kinks [2] and to the residual energy with similar power-law scaling laws. In this paper, we do not attempt to reproduce critical exponents for c-bits or p-bits, but simply observe the power-law

dependence of c-bits and p-bits with respect to α in 2D ferro-Ising and 3D spin glass models.

II. Supplementary Note 2

Algorithm 1: Adaptive Parallel Tempering Preprocessing with p-bits

Input: energy variance tolerance, number of chains, number of sweeps, number of samples, constant parameter α
Output: problem-specific β schedule for parallel tempering
$i \leftarrow 1$;
$\beta_1 \leftarrow 0.5$;
Initialize chains to random spins;
while *energy variance* σ_E *is greater than tolerance* **do**
  for *each chain in parallel* **do**
    Simulate p-bits using Eq. 1 and 2 for the specified number of Monte Carlo sweeps at constant inverse temperature β_i;
    Calculate variance σ_E of energies sampled at the end of the simulation;
    Save the spin state $\{m\}$ of each chain as the initial condition for the next iteration;
  end
  Calculate the mean energy variance across chains, $\langle \sigma_E \rangle$;
  $\beta_{i+1} \leftarrow \beta_i + \frac{\alpha}{\langle \sigma_E \rangle}$;
  $i \leftarrow i + 1$;
end

We use the following parameters for APT preprocessing: 100 parallel chains, 10^4 Monte Carlo sweeps, 1000 samples (taken at the end of 10^4 sweeps), and $\alpha = 1.25$. Energy variance tolerance is equal to half of the smallest magnitude weight J_{ij}, and the initial value of β is set to 0.5. We have used a similar preprocessing method in our previous work [3].

III. Supplementary Note 3

To derive our time-to-solution performance metric (Eq. 19), we start from:

$$[1 - p(Nt_a)]^k = 1 - 0.99 \tag{S9}$$

where $p(Nt_a)$ is the probability of finding the solution using N system replicas each running for time t_a. The left-hand side of the equation represents the probability of failing k times in a row, and the right-hand side is the complement of the desired

success probability, 0.99. We solve for k through algebraic manipulation. The time-to-solution is then:

$$Nt_a \times k \tag{S10}$$

which yields our definition of time-to-solution (Eq. 19). This can be thought of as the total computational time of the algorithm, Nt_a, multiplied by the average number of attempts before success.

Supplementary References

[1] Del Campo, A. & Zurek, W. H. Universality of phase transition dynamics: Topological defects from symmetry breaking. *International Journal of Modern Physics A* **29**, 1430018 (2014).

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[3] Nikhar, S., Kannan, S., Aadit, N. A. *et al.* All-to-all reconfigurability with sparse and higher-order ising machines. *Nature Communications* **15**, 8977 (2024).