

Supplementary Information:

Nested Chirality and Multimode Transfer Around Exceptional Points

Driven by External Forces in Electric Circuits

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Supplementary Note 1. Eigenvectors and the range of parameters.

The Hamiltonian of the 2D non-Hermitian HOTI model can be written as

$$H_{\mathbf{k}} = \sum_{i=1}^4 d_i(\mathbf{k}) \Gamma_i, \quad (\text{S1})$$

where

$$\begin{aligned} d_1(\mathbf{k}) &= (q - \lambda) \sin(k_y/2) - i\gamma \cos(k_y/2), \\ d_2(\mathbf{k}) &= -(q + \lambda) \cos(k_y/2) - i\gamma \sin(k_y/2), \\ d_3(\mathbf{k}) &= -(q - \lambda) \sin(k_x/2) + i\gamma \cos(k_x/2), \\ d_4(\mathbf{k}) &= (q + \lambda) \cos(k_x/2) + i\gamma \sin(k_x/2). \end{aligned} \quad (\text{S2})$$

The twofold degenerate energy bands can be expressed as $\pm\varepsilon = \pm\sqrt{d_1^2 + d_2^2 + d_3^2 + d_4^2}$.

The right eigenvectors of the model read

$$\begin{aligned} |\psi_1\rangle &= \frac{1}{\sqrt{2\varepsilon}}(-id_1 - d_2, id_3 - d_4, 0, \varepsilon)^T, \\ |\psi_2\rangle &= \frac{1}{\sqrt{2\varepsilon}}(-id_3 - d_4, -id_1 + d_2, \varepsilon, 0)^T, \\ |\psi_3\rangle &= \frac{1}{\sqrt{2\varepsilon}}(id_1 + d_2, -id_3 + d_4, 0, \varepsilon)^T, \\ |\psi_4\rangle &= \frac{1}{\sqrt{2\varepsilon}}(id_3 + d_4, id_1 - d_2, \varepsilon, 0)^T. \end{aligned} \quad (\text{S3})$$

Here, $|\psi_1\rangle$ and $|\psi_2\rangle$ correspond to band $-\varepsilon$, while $|\psi_3\rangle$ and $|\psi_4\rangle$ correspond to band $+\varepsilon$. The corresponding left eigenvectors are

$$\begin{aligned} \langle\varphi_1| &= \frac{1}{\sqrt{2\varepsilon}}(id_1 - d_2, -id_3 - d_4, 0, \varepsilon), \\ \langle\varphi_2| &= \frac{1}{\sqrt{2\varepsilon}}(id_3 - d_4, id_1 + d_2, \varepsilon, 0), \\ \langle\varphi_3| &= \frac{1}{\sqrt{2\varepsilon}}(-id_1 + d_2, id_3 + d_4, 0, \varepsilon), \\ \langle\varphi_4| &= \frac{1}{\sqrt{2\varepsilon}}(-id_3 + d_4, -id_1 - d_2, \varepsilon, 0). \end{aligned} \quad (\text{S4})$$

It is obvious that the eigenvectors satisfy the biorthogonal relationship $\langle\varphi_i|\psi_j\rangle = \delta_{ij}$ for $i, j = 1, 2, 3, 4$.

Now we discuss the appropriate range of parameters q , γ and λ that allows the encircling of the EP in the g - ω space.

Along the diagonal path $k_x = k_y = k$, We take $\omega = \lambda \sin k$ and $g = \gamma + \lambda \cos k$, then the energy bands along the diagonal path can be expressed as $E = \pm\sqrt{2(\omega + i\gamma)^2 + 2g^2 + 2q^2 - 2\gamma^2 + 4(q - \gamma)(g - \gamma)}$. The EPs ($E = 0$) exist on the g - ω plane along the g axis ($\omega = 0$) at $g_1 = -q$ and $g_2 = 2\gamma - q$, which are represented by the yellow dots shown in Fig. S1. Meanwhile, the closed path of k from 0 to 2π forms a circle centered at $(\gamma, 0)$ with λ being the radius on the g - ω plane, and the circle intersects the g axis at $g_3 = \gamma - \lambda$

and $g_4 = \gamma + \lambda$, which are represented by the blue dots shown in Fig. S1. Here, we suppose that all parameters are positive, so that $g_1 < g_2$ and $g_3 < g_4$.

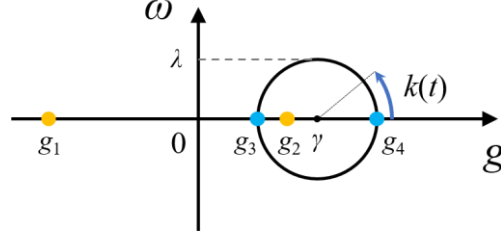


Figure S1 Exceptional points (EPs) and the path of momentum. The two EPs g_1 and g_2 are represented by the yellow dots on the g axis on the g - ω plane. The path of momentum intersects the g axis at g_3 and g_4 , which are marked by the blue dots.

If $g_1 < g_3 < g_2 < g_4$, the circle can just surround the EP g_2 , which is satisfied for $|q - \gamma| < \lambda < q + \gamma$. If $\lambda > q + \gamma$, the circle can surround the two EPs g_1 and g_2 . In such conditions, the encircling processes have the same consequences but experience more NATs, which can also exhibit the nested chiral behavior introduced in the main text. We do not go into details here, because a similar comparison is discussed in Ref. [1].

Now we calculate the eigenstates at the EPs. We set $k_x = k_y = 0$ and $q = \gamma - \lambda$. We found that the four eigenstates can be expressed as $v_1 = (0, 0, -1, 1)^T$, $v_2 = (0, 1, 0, 0)^T$ and $v_3 = v_4 = (0, 0, 0, 0)^T$. Since there are two defective states, the EPs are twofold degenerate second-order EPs rather than fourth-order EPs.

Supplementary Note 2. Descriptions of the sequence and results of the state transitions.

In the main text, we calculate the numerical results of the state transitions when encircling the EP. In this section, we explain the sequence and results of the state transitions. We consider a counterclockwise encircling. At the beginning the states on the lower Riemann surface ($|\psi_1\rangle$ and $|\psi_2\rangle$) lose and the states on the upper Riemann surface ($|\psi_3\rangle$ and $|\psi_4\rangle$) gain. So, if we choose a state on the lower Riemann surface as the initial state, the evolutionary state generally

loses at the beginning. However, states on the upper Riemann surface are excited due to the non-adiabatic couplings (NACs) between the states on the lower Riemann surface and the upper Riemann surface, and the excited gain states gradually increase and eventually exceed the loss states, forming a NAT process [2]. Meanwhile, because the NACs between different states are not the same, there is a certain tendency when the NAT occurs. The NAC can be calculated as [3-5]

$$A_{ij} = \left\langle \varphi_i \left| \frac{\partial}{\partial t} \right| \varphi_j \right\rangle$$

$$= \frac{1}{2dt} \left(\langle \varphi_i(t) | \varphi_j(t+dt) \rangle - \langle \varphi_i(t+dt) | \varphi_j(t) \rangle \right). \quad (\text{S5})$$

Here, we calculate the NAC between $|\psi_1\rangle$ and $|\psi_3\rangle$ ($|\psi_4\rangle$), as shown in Fig. S2. It can be seen that during the encircling process, the NAC between $|\psi_1\rangle$ and $|\psi_3\rangle$ (A_{13}) is always zero and the NAC between $|\psi_1\rangle$ and $|\psi_4\rangle$ (A_{14}) is non-zero. As a result, if we choose $|\psi_1\rangle$ as the initial state, the first NAT emerges between $|\psi_1\rangle$ and $|\psi_4\rangle$ instead of that between $|\psi_1\rangle$ and $|\psi_3\rangle$ after some time evolution, which corresponds to the results in Fig. 2 of the main text.

At the branch cut of the Riemann surfaces, the real parts of $|\psi_2\rangle$ and $|\psi_4\rangle$ coincide and the real parts of $|\psi_1\rangle$ and $|\psi_3\rangle$ also coincide. So $|\psi_4\rangle$ flips to $|\psi_2\rangle$ instead of $|\psi_1\rangle$ after crossing the branch cut. Other results with different initial states and different encircling directions can also be analyzed by similar methods.

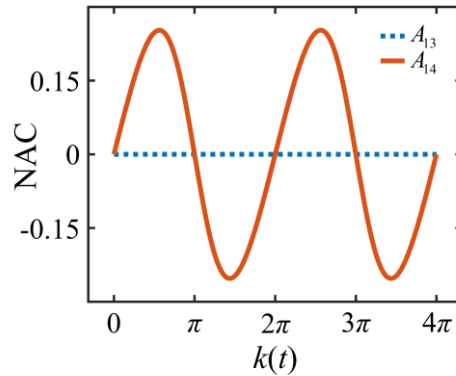


Figure S2 Non-adiabatic coupling (NAC) between $|\psi_1\rangle$ and $|\psi_3\rangle$ ($|\psi_4\rangle$). The blue dotted line represents the NAC between $|\psi_1\rangle$ and $|\psi_3\rangle$, which is always zero during the encircling process. The red solid line represents the NAC between $|\psi_1\rangle$ and $|\psi_4\rangle$, which is non-zero during the encircling process.

Supplementary Note 3. Negative impedance converter with current inversion and negative resistance module.

In this section, we introduce the structure and the function of the negative impedance converter with current inversion (INIC) and the negative resistance module (NRM). The INIC between nodes j and k consists of one operational amplifier (OpAmp) and three resistors, as shown in Fig. S3. The two auxiliary resistors in the positive and negative feedback loops of the OpAmp take the same resistances of R_a , while the third resistor (effective resistor) takes the resistance of R_{jk} .

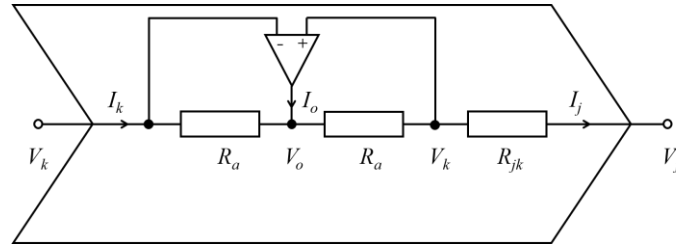


Figure S3 The structure of a negative impedance converter with current inversion (INIC).

An INIC consists of one operational amplifier and three resistors, which forms inverted currents at the two terminals.

When the OpAmp works in the linear region, the currents flowing into the input terminals of the OpAmp are approximately zero, and the potential at the inverting input terminal equals to that of the non-inverting input terminal, which is the voltage at node k (V_k). Thus, we can calculate the currents flowing into the nodes j and k as

$$I_j = \frac{V_k - V_j}{R_{jk}} = \frac{V_o - V_k}{R_a}, \quad (\text{S6})$$

$$I_k = \frac{V_k - V_o}{R_a}. \quad (\text{S7})$$

We find that $I_k = -I_j$, which means the current at the terminal near node k of the INIC is inverted. Thus, the INIC contributes a negative resistance $-R_{jk}$ to the current flowing into the node k , and a resistance R_{jk} to the current flowing into the node j .

An NRM is an extension of an INIC. The NRM between nodes j and k consists of two OpAmps and five resistors, as shown in Fig. S4. The four auxiliary resistors in the positive and negative feedback loops of the two OpAmps take the same resistances of R_a , and the effective resistor takes the resistance of R_{jk} .

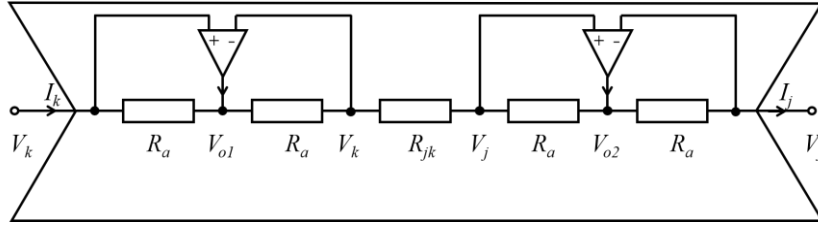


Figure S4 The structure of a negative resistance module (NRM). An NRM consists of two operational amplifiers and five resistors, which contributes a negative resistance to both terminals.

We can write the currents flowing into the nodes j and k as

$$I_j = \frac{V_{o2} - V_j}{R_a}, \quad (\text{S8})$$

$$I_k = \frac{V_k - V_{o1}}{R_a}. \quad (\text{S9})$$

When the OpAmp works in the linear region, we have

$$\frac{V_{o2} - V_j}{R_a} = \frac{V_j - V_k}{R_{jk}} = \frac{V_k - V_{o1}}{R_a}. \quad (\text{S10})$$

We find that $I_j = I_k = (V_j - V_k)/R_{jk}$, which means the NRM contributes a negative resistance $-R_{jk}$ to both nodes j and k . Using a combination of INICs and NRMs, the designed circuit can correspond to the non-Hermitian model in the main text.

Supplementary Note 4. Two kinds of turn of the real-space motion of the wave packet's center-of mass.

In this section, we discuss the turns of the real-space motion of the wave packet's center-of mass. The electron velocity of a lattice model under an applied force satisfies the semiclassical equation as $d\mathbf{r}/dt = dE/d\mathbf{k} - \mathbf{F} \times \eta^\dagger \mathbf{B} \eta$, where $\mathbf{B} = \nabla_{\mathbf{k}} \times \mathbf{A} - i\mathbf{A} \times \mathbf{A}$ denotes the non-Abelian Berry curvature. The second term of the right part of the above equation is the anomalous velocity perpendicular to $\mathbf{F}$, which leads to the transverse Hall drift. Here, we focus on the parallel velocity $\mathbf{v} = dE/d\mathbf{k}$.

According to the expression of the eigenvalue E , we calculate the relationship between the parallel velocities and the momentum during the encircling processes. Figures S5a and S5b show the parallel velocities (real parts) in counterclockwise and clockwise encircling processes, respectively. Here, v_u (blue solid lines) and v_l (red dashed lines) represent the parallel velocities of the states in upper and lower bands, respectively. The brand bands represent the transform of the largest occupied states. It can be seen that v_u and v_l are symmetric with respect to zero. At $k = \pi$ and 3π ($-\pi$ and -3π), v_l and v_u flip over to each other. At the same time, the wave packet experiences adiabatic state flips so that the sign of the parallel velocity does not change. At $k = 2\pi$ (-2π), the sign of the parallel velocity changes, which corresponds to the BO phenomena represented by the green boxes in Figs. 4b and 4c in the main text.

If a NAT happens, the wave packet flips from the upper (lower) band to the lower (upper) band, and the parallel velocity is reversed, see the red boxes in Fig. S5a, which corresponds to another kind of turn for the wave packet's motion represented by the red boxes in Figs. 4b and 4c in the main text. The latter is unique to non-Hermitian systems and is distinct from the BO phenomenon in Hermitian systems.

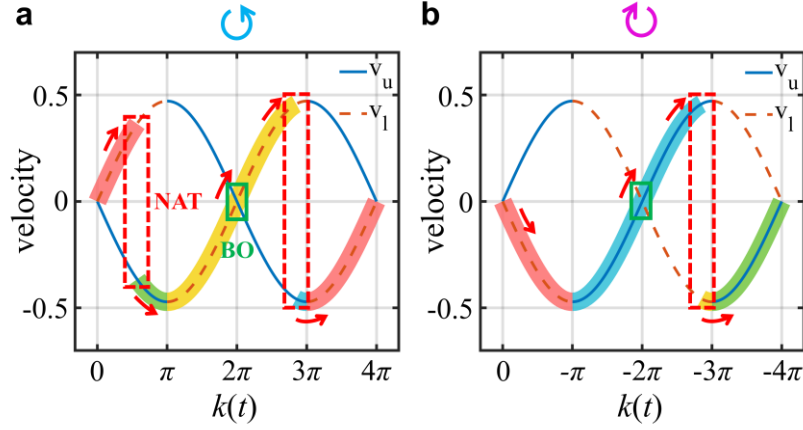


Figure S5 Parallel velocities in the encircling processes. **a** The real parts of the parallel velocities for states in the upper (blue solid lines) and the lower (red dashed lines) bands in counterclockwise encircling. The brand bands represent the transform of the largest occupied states, where the red, yellow, blue and green bands represent $|\psi_1\rangle$, $|\psi_2\rangle$, $|\psi_3\rangle$ and $|\psi_4\rangle$, respectively. Here the red arrows indicate the change of the velocity for the wave packet. The red and green boxes indicate the NAT processes and the BO phenomena, respectively. **b** The parallel velocities in clockwise encircling.

Supplementary Note 5. Other experimental settings and results.

In this section, we give more details about the experimental settings and the results of other initial states. In the circuit network we made, we use the two-gate operational amplifier model LT1013 to implement the INICs and NRMs. To achieve the intracell hopping amplitudes of 1 ± 0.8 , we choose surface-mounted device (SMD) resistors of $1 \text{ k}\Omega$ as the effective resistors of the intracell INICs and SMD resistors of $1.24 \text{ k}\Omega$ as the effective resistors of the NRMs. The intercell hopping amplitude of 0.3 is realized by INICs with SMD resistors of $3.32 \text{ k}\Omega$ as the effective resistors. Moreover, we choose appropriate effective resistances of the INICs for the specific on-site potentials and the grounding parts. Here, the connection directions of the INICs for the on-site potentials can be changed to realize different directions of the applied force. The feedback resistances of the INICs and the NRMs are selected to be the same as the effective resistances. The tolerance of all the resistors is less than 0.1% . In addition, we choose thin film

capacitors of 1 μF as the grounding capacitors. All the capacitors are manually selected to the theoretical capacitances within a tolerance of 1%.

At the beginning of the experiment, all nodes are connected to the DC power supplies through relays. In the experiment for the counterclockwise encircling of the EP, we set the real nodes voltages of each unit cell to be (0.24, 1.00, 0.00, 0.70) V, and the imaginary nodes voltages all to be 0 V, respectively. While for the clockwise encircling of the EP, as the simulation results show that the evolutionary voltages gradually increase to about 10^2 times of the initial voltages during the clockwise encircling, we set the real nodes voltages of each unit cell to be (0.02, 0.08, 0.00, 0.06) V, and the imaginary nodes voltages all to be 0 V, respectively, to avoid the voltages exceeding the maximum outputs of the operational amplifiers (about 10V). With the above settings, we can observe the dynamical encircling of the EP on the circuits experimentally.

In the main text, we have discussed the experimental observation of encircling the EP with initial state $|\psi_1\rangle$. Here, we give the experimental results of encircling the EP with the other three initial states. We use the same PCBs as the experiments for $|\psi_1\rangle$. The only change to be made is that the initial excitations of the voltages should correspond to $|\psi_{r,2}\rangle$, $|\psi_{r,3}\rangle$ and $|\psi_{r,4}\rangle$, which are the real-space circuit wave packet corresponding to $|\psi_2(k=0)\rangle$, $|\psi_3(k=0)\rangle$ and $|\psi_4(k=0)\rangle$, respectively.

The experimental results in counterclockwise and clockwise encircling with different initial states are show in Fig. S5 and S6, respectively. It can be seen that the experimental results fit the theoretical simulations. Combining the experimental results with initial state $|\psi_1\rangle$, the nested chiral behavior is observed through our circuit networks.

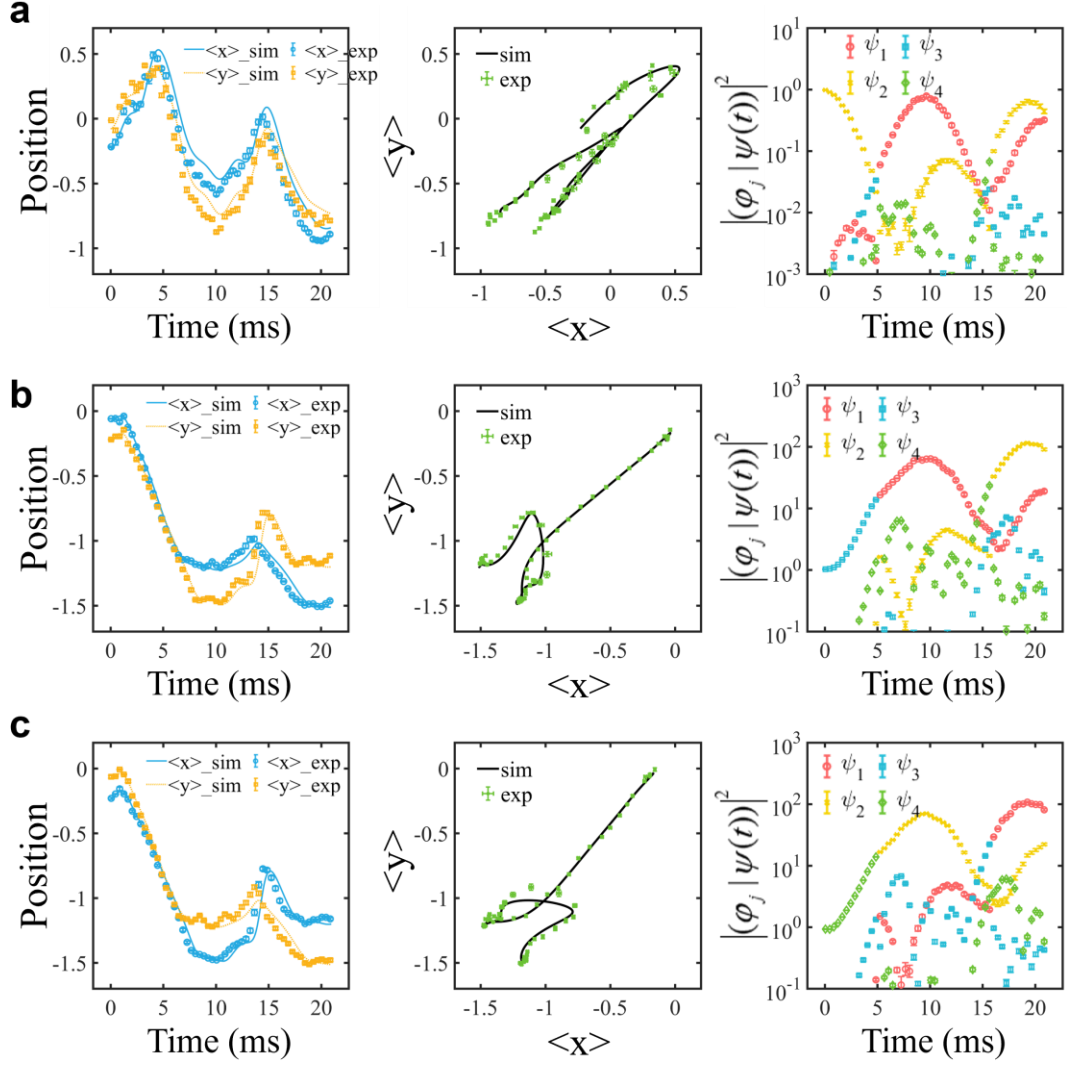


Figure S6 Experimental results of counterclockwise encircling with different initial states.

a, b and **c** Experimental results of center-of mass evolution, real-space trajectory and states populations in counterclockwise encircling with initial states $|\psi_2\rangle$, $|\psi_3\rangle$ and $|\psi_4\rangle$, respectively. Here, the abbreviation "sim" stands for simulation and the abbreviation "exp" stands for experiment.

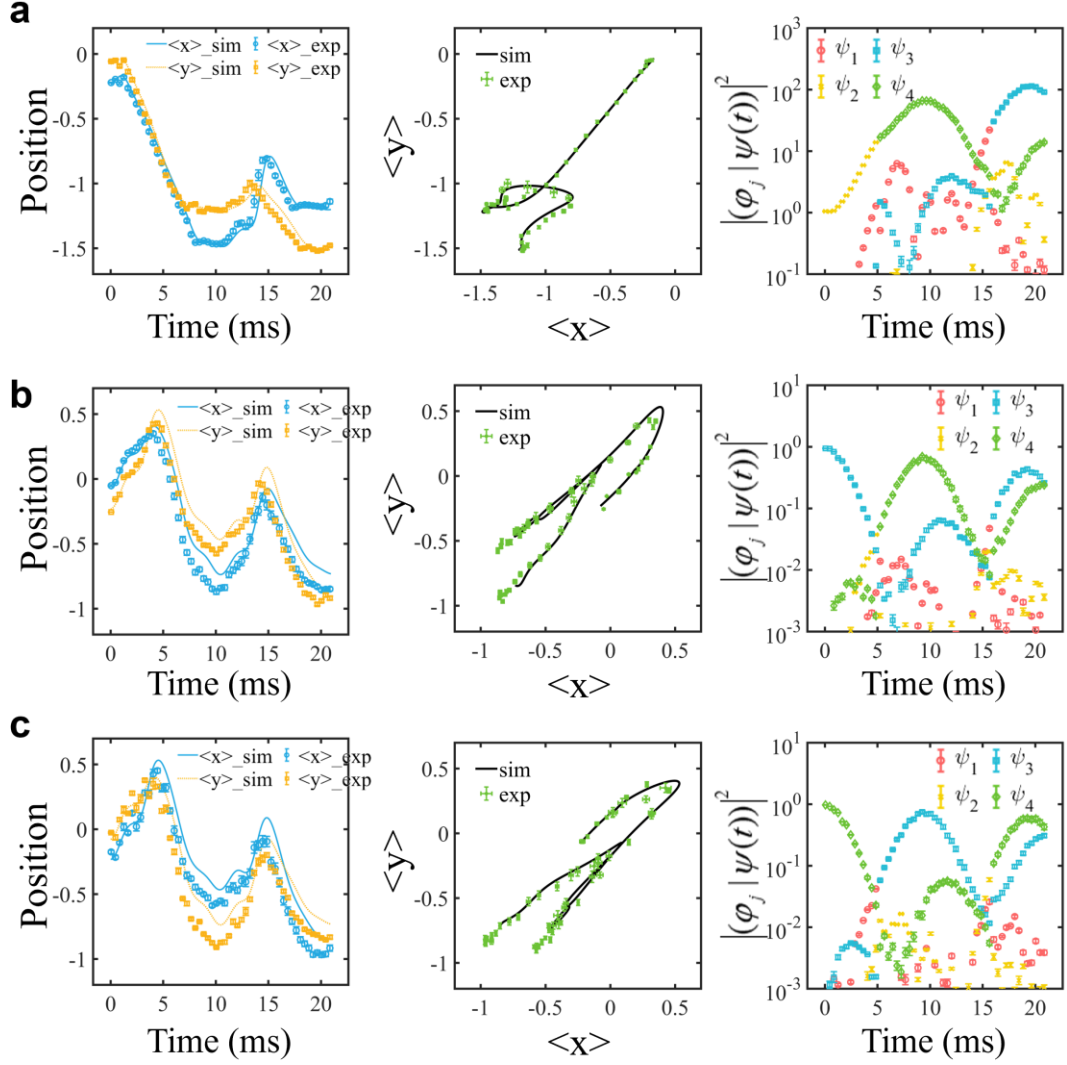


Figure S7 Experimental results of clockwise encircling with different initial states. **a, b** and **c** Experimental results of center-of-mass evolution, real-space trajectory and states populations in clockwise encircling with initial states $|\psi_2\rangle$, $|\psi_3\rangle$ and $|\psi_4\rangle$, respectively. Here, the abbreviation "sim" stands for simulation and the abbreviation "exp" stands for experiment.

Supplementary Note 6. Robustness of the encircling processes under different degrees of disorder.

In this section, we study the influence of the disorder of the circuit elements and discuss the robustness of the encircling processes. Figure S8 shows the simulation results for the encircling processes on the circuits with different degrees of disorder. The tolerances of the

circuit elements are set to 5%, 10% and 15%, respectively. It can be seen that when the circuit elements are under a limited disorder, e.g., 5% and 10%, the evolutions of the real-space center-of mass are basically consistent with the ideal conditions. Meanwhile, the variations of the largest occupied state also match the theoretical results. These results show the good robustness of the encircling processes in the circuit system. However, if the disorder of the circuit elements is too large, e.g., 15%, the nested chiral behavior of the encircling processes behavior cannot be displayed.

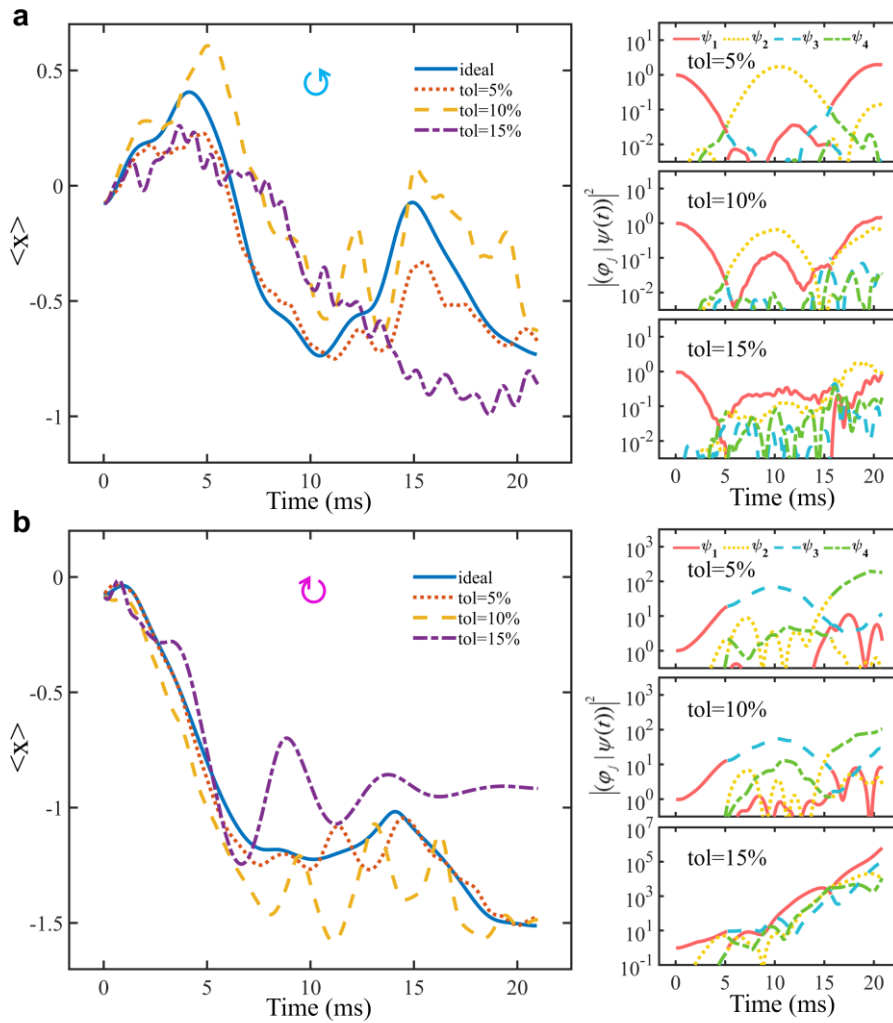


Figure S8 Simulation results under different degrees of disorder. **a** Simulation results of center-of mass evolution and states populations in counterclockwise encircling with tolerances of the circuit elements are set to 5%, 10% and 15%, respectively. **b** Simulation results of center-of mass evolution and states populations in clockwise encircling.

Supplementary References

1. Zhang, X. L., Chan, C.T. Dynamically encircling exceptional points in a three-mode waveguide system. *Commun. Phys.* **2**, 63 (2019).
2. Uzdin, R., Mailybaev, A. & Moiseyev, N. On the observability and asymmetry of adiabatic state flips generated by exceptional points. *J. Phys. A: Math. Theor.* **44**, 435302 (2011).
3. Hammes-Schiffer, S. & Tully, J. C. Proton transfer in solution: Molecular dynamics with quantum transitions. *J. Chem. Phys.* **101** (6), 4657–4667 (1994).
4. Akimov, A. V. Nonadiabatic Molecular Dynamics with Tight-Binding Fragment Molecular Orbitals. *J. Chem. Theory Comput.* **12** (12), 5719–5736 (2016).
5. Milburn, T. J. et al. General description of quasiadiabatic dynamical phenomena near exceptional points. *Phys. Rev. A* **92**, 052124 (2015).