# Peer Review File

## Dynamical topology of chiral and nonreciprocal state transfers in a non-Hermitian quantum system

Corresponding Author: Professor Le Luo

**This file contains all reviewer reports in order by version, followed by all author rebuttals in order by version.**

Version 0:

Reviewer comments:

Reviewer #1

(Remarks to the Author)
This article presents an experimental study on dynamical evolution of a single qubit governed by a non-Hermitian Hamiltonian with PT symmetry. By winding around the exceptional point in different directions, nonreciprocal state transfer has been witnessed. The dynamics can be described by an invariant, which is called a dynamical topological invariant. The results discussed in this work are new and interesting. I would like recommend the paper to be published in Communications Physics if the authors can address the following points.

1. From a theoretical point of view, what is the role of PT symmetry assumed by the authors? The trapped ion is a purely disspative system and is only passive PT symmetric. That is, it can only be made PT-symmetric with an extra diagonal term. What if the extra term is not included? I believe the nonreciprocal state transfer should still be observed if one works direction on the original passive PT system. Is this correct? Can one still define a dynamical topological invariant? Is it of the same form as the one discussed here?

2. My major concern of this work is the dynamical process simulated by a trapped ion. Since the ion qubit is purely lossy, the particle number will be very small for long enough evolution time. The authors use a so-called piecewise strategy by slicing the evolution period into many small intervals. Within each interval, they initialize the qubit to the state predicted by theoretical calculation, and let it evolve only within that interval. That is, the evolution is not actually realized in a physical system, but is only illustrated, or more precisely, imitated by a sequence of fine-tuned snapshots. My question is that are the phenomena discussed here can be observed in any realistic quantum systems? The small particle number at long evolution time should be a very common challenge. If so, in which case the findings here can be of any practical value?

3. Another question is about the validity of non-Hermitian Hamiltonian. It is well known that the difference between non-Hermitian Hamiltonian and Lindblad equation will become more significant as time evolves. It may not be a severe problem here, since the real evolution is only carried within a short time interval. But it will be of great interest to see how robust the dynamical topological invariant can be if one works in the more accurate Lindblad equation framework.

4. Words and labels in many figures are too small to be read.

5. There are no solid or dashed trajectories in Fig. 1(a) as mentioned in the caption.

Reviewer #2

(Remarks to the Author)
This is a review report of the manuscript “Dynamical topology of chiral and nonreciprocal state transfers in a non-Hermitian quantum system” by Pengfei Lu, Yang Liu et al. (COMMSPHYS-24-1114). The authors experimentally investigate a time-dependent, non-Hermitian trapped-ion system exhibiting exceptional points (EPs) on the eigenvalue Riemann sheets. By dynamically encircling the EPs, they observed intriguing topological chiral and nonreciprocal dynamics in both \$\mathcal{PT}\$-symmetric and \$\mathcal{APT}\$-symmetric Hamiltonians. Their results demonstrate that these dynamics are topologically robust against perturbations and are protected by a topological invariant-the dynamical vorticity-through the use of a parallel transported eigenbasis.

Based on the comprehensive experimental results and well-formulated conclusions, I believe this work makes a significant contribution to the field of open quantum systems. It offers valuable insights into novel effects induced by EPs. Therefore, I recommend this paper for publication in Communications Physics, pending the authors' response to the following questions:

1)In Fig 3 (a), with a fixed $\tau_{crit}=11.8\mu s$, could the authors please explain why the contour lines of $\tau_{crit}/T$ (note: incorrectly labeled as $t_{crit}/T$ in the figure ) appear irregular and jagged in the vertical direction (for a fixed encircling period $T$)? Are these contours derived from simulation or experimental results?

2)Regarding the instantaneous eigenstates $\ket{\alpha_A(t)}$ and $\ket{\beta_A(t)}$, do they represent the ideal adiabatic case where no DNAT occurs? For instance, in the trajectory 1 in Fig. 2, when the encircling angle reaches $2\pi$ at time $t$, does $\ket{\beta_A(t)}$ transition to $\ket{\alpha_A(0)}$? If this is the case, I would kindly suggest that the authors clarify this point when they first introduce these instantaneous eigenstates.

3)As demonstrated in the abstract and introduction, in the absence of DNATs, the evolutionary state is expected to acquire a geometric phase of $\pi$ after encircling the EP twice, which is an indication of novel topology. However, I feel that the authors have not clearly demonstrated whether this geometric phase is reflected in their experimental results, or how it is manifested. Additionally, does the geometric phase of $\pi$ correspond to the half-integer dynamic vorticity $\mathcal{V_D}$ as defined by Eq. (4)? If so, could the authors please clarify why this is the case?

4)The authors demonstrated that the vorticity $\mathcal{V}_D$ equals to $\pm 1/2$ for trajectories 2 and 1, confirming the chiral state transfer. My question is: for the trajectories 5 and 6, where chiral symmetry is broken, what are the values of $\mathcal{V}_D$? Furthermore, as introduced in Ref [13], chiral transfer can still occur without encircling an EP, in which case, as demonstrated in this paper, $\mathcal{V}_D$ becomes 0. Does this mean that the topological invariant $\mathcal{V}_D$ fails in such a scenario?

Reviewer #3

(Remarks to the Author)
In this paper, authors discuss dynamical circling of exceptional points and experimentally observed chiral state transfers. The results sound reasonable, and the data are properly presented. However, the following points should be clarified before judging whether the manuscript warrants the publication.

a) As discussed in "Chiral and nonreciprocal state transfers", dissipation-induced nonadiabatic transitions (DNATs) result in chiral state transfers. However, it is unclear what its mechanism is and when it occurs. This point should be clarified.

b) The authors claim that Eq. (4) is their new result. However, it is essentially the same as the vorticity defined in Ref. 20 [PRL 120 146402 (2018)]. The authors should explain the difference.

c) SNAT before "Methods" is not defined.

Version 1:

Reviewer comments:

Reviewer #1

(Remarks to the Author)
The authors have addressed all my questions and concerns with solid arguments and detailed calculations. The manuscript has also been revised properly. I thus recommend the paper to be accepted by Communications Physics.

Reviewer #2

(Remarks to the Author)
Thank you for your detailed and thorough responses. I believe the changes and explanations you provided have effectively addressed my initial concerns.

However, based on your reply to my question (4), where you stated that 'the vorticity V_D is not used to confirm the chiral state transfer' and that 'even when V_D=0, this invariant can indicate the robustness of the chiral state transfer,' it appears that the chiral state transfers can occur both when V_D=0 and when V_D≠0. Therefore, the statement on the bottom of Page 12, '... confirming chiral and nonreciprocal state transfers are topologically protected.' may not be entirely appropriate in this context. I suggest further clarifying this point in the main article to avoid potential misunderstandings for readers.

Overall, with this clarification, I believe the manuscript will be suitable for publication.

Reviewer #3

(Remarks to the Author)
The authors addressed my questions properly. Therefore, I recommend the publication in Comm Phys.

# Responses to Reviewers' Questions and Comments

In the following, please find a detailed reply to the reviewers' comments. The review reports are reproduced below in blue, and our point-by-point responses are in black.

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Reviewer #1

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This article presents an experimental study on dynamical evolution of a single qubit governed by a non-Hermitian Hamiltonian with PT symmetry. By winding around the exceptional point in different directions, nonreciprocal state transfer has been witnessed. The dynamics can be described by an invariant, which is called a dynamical topological invariant. The results discussed in this work are new and interesting. I would like recommend the paper to be published in Communications Physics if the authors can address the following points.

Reply: We appreciate the reviewer's appreciation of this work.

1. From a theoretical point of view, what is the role of PT symmetry assumed by the authors? The trapped ion is a purely dissipative system and is only passive PT symmetric. That is, it can only be made PT-symmetric with an extra diagonal term. What if the extra term is not included? I believe the nonreciprocal state transfer should still be observed if one works direction on the original passive PT system. Is this correct? Can one still define a dynamical topological invariant? Is it of the same form as the one discussed here?

Reply: We thank the reviewer for the comments and questions. First, we explain the role of PT symmetry in our work. In the study of non-Hermitian quantum systems, PT symmetry provides a unique framework that real eigenvalues exist under certain conditions [Phys. Rev. Lett. 80. 5243]. This makes PT-symmetric Hamiltonians valuable in modeling dissipative quantum dynamics, allowing us to investigate physics in open quantum systems, such as phase transitions from PT-symmetric to broken PT regions [Nat. Commun 10, 855 (2019)], quantum speed limits [New J. Phys. 26 013043], and temporal correlations induced by dissipation [Phys. Rev. A.109.042205]. Based on these previous experimental works, we believe that the PT-symmetric Hamiltonians provide a model system to investigate the dynamically encircling of the EP.

Second, we agree with the reviewer that the trapped ion is a purely dissipative system and is only passive PT symmetric. PT-symmetric can be constructed by introducing an extra diagonal term. Importantly, the absence of this diagonal term does not affect the key findings regarding nonreciprocal or chiral state transfer. As illustrated in Fig. 2, the eigenvalue Riemann surface and the trajectories of state transfer remain unchanged without this extra diagonal term. So your guess is correct.

Third, the extra diagonal term does not alter the topology of the eigenvalue Riemann surface. Consequently, the definition and form of the dynamical topological invariant remain applicable to both PT-symmetric and passive PT-symmetric Hamiltonians. We have added a clarifying sentence to the "Discussion" section, which can be found in paragraph 3 on page 12, as follows "It is also noted that the definition and form of Eq. (4) apply to both PT-symmetric and passive PT-symmetric Hamiltonians, since introducing an extra diagonal term does not alter the basic topology of the dynamics."

2. My major concern of this work is the dynamical process simulated by a trapped ion. Since the ion qubit is purely lossy, the particle number will be very small for long enough evolution time. The authors use a so-called piecewise strategy by slicing the evolution period into many small intervals. Within each interval, they initialize the qubit to the state predicted by theoretical calculation, and let it evolve only within that interval. That is, the evolution is not actually realized in a physical system, but is only illustrated, or more precisely, imitated by a sequence of fine-tuned snapshots. My question is that are the phenomena discussed here can be observed in any realistic quantum systems? The small particle number at long evolution time should be a very common challenge. If so, in which case the findings here can be of any practical value?

Reply: Thanks for the reviewer's comment. The pure loss nature does lead to a very small state population for long evolution time. It is why we adopt the piecewise strategy to slice the evolution period into many small intervals. In Figure 5 of the manuscript, we adjust the size of $N$ to test the effectiveness of the piecewise strategy. The experimental results for $N$=100 closely align with the numerical simulation. Each segment undergoes a 2.5 us evolution (total encircling period of 250 us for $N$ =100), which leads to approximately 24% change in the state population, as shown in Figure 1 below. Such a population change is obvious.

We want to emphasize that, all of the previous experiments for the parametric encirclement [Phys. Rev. Lett. 86, 787, Phys. Rev. Lett. 103, 134101, Phys. Rev. Lett. 127, 034301, Nature 554, 526] have used a "stroboscopic" approach, where systems are constructed at various parametric points along a close loop and then measured separately.

There is no population change at each parametric point because of stationary measurements. Compared to these previous experiments, our experiments show a significant population change, thus it can NOT be simply viewed as a sequence of fine-tuned snapshots.

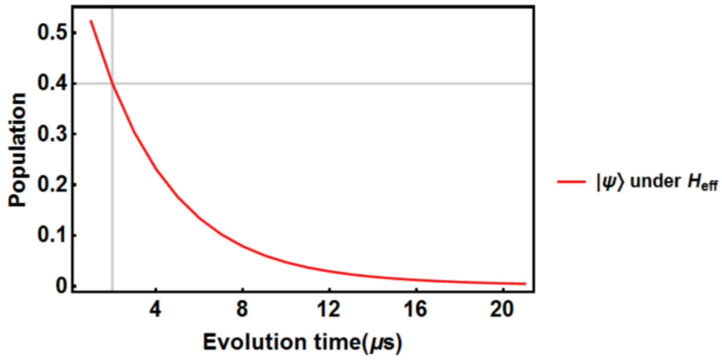


Figure 1. The time evolution of eigenstate under the effective Hamiltonian. The simulation parameters match the values used in our experiments.

In terms of your question that if the chiral and nonreciprocal behaviors discussed here can be observed in other realistic quantum systems, our answer is “YES”. Typical examples include superconducting circuits [Phys. Rev. Lett. 128. 160401], nitrogen-vacancy centers [Phys. Rev. Lett. 126. 10506], and cold atoms [Nat. Phys. 18, 385 (2022)].

Compared to these experiments, the method that we use to observe chiral and nonreciprocal state transfers has a significant advantage: we could implement arbitrary encircling radius and speed in a real dissipative system. Experiments in [Phys. Rev. Lett. 128. 160401, Nat. Phys. 18, 385 (2022)] use a real dissipative system, but they need to design larger encircling radius and faster speed to avoid very small state population. Experiment in [Phys. Rev. Lett. 126, 10506] could implement arbitrary encircling radius and speed, but their system is not a real dissipative system. They dilate the PT-symmetric Hamiltonian into a Hermitian form in a higher-dimensional Hilbert space.

In terms of your question about the practical value for small particle number, this is the fact in many dissipative systems. But there are many solutions to let the system useful. In classical systems of similar non-Hermitian Hamiltonians, the issue can be avoided by introducing the gain to compensate the dissipation. Along this line, many studies in classical system have shown great potential for practical applications, such as enhancing the sensitivity of sensors [Nature 548, 192 (2017)], developing optical isolators [Nat. Mater. 18, 783 (2019)], and enabling novel applications in laser control and mode switching [Nat. Nanotechnol. 18, 706 (2023)], among others.

In quantum systems, avoiding and compensating for particle number loss is much more challenging compared to classical systems. In order to address this issue, the dilation method [Science 364, 878 (2019), Phys. Rev. Lett. 126. 10506], which is a mathematical technique mapping a non-Hermitian Hamiltonian to a higher-dimensional Hermitian Hilbert space, could be used. Therefore, studies of non-Hermitian dynamics have practical values to reveal complex dynamic structures of the Hermitian systems.

Another method is the post-selection technique [Arxiv:2304.12413], which enables the realization of PT-symmetric Hamiltonians in quantum systems by conditioning the system on specific measurement outcomes. Using this method, we could let this system have practical values even with very small particle number. For example, chiral behaviors in quantum system also demonstrated practical value in applications such as chiral control of quantum states utilizing spin-orbit coupling [Nat. Phys. 18, 385 (2022)], enhancement of quantum heat engines through encircling a Liouvillian exceptional point (LEP) [Phys. Rev. Lett. 130. 110402], and chiral thermodynamic effects near an LEP [Light: Science & Applications 13, 143 (2024)]. These experimental breakthroughs will pave the way for testing theoretical predictions and discovering new non-Hermitian phenomena and bridging the gap between theory and practice.

3. Another question is about the validity of non-Hermitian Hamiltonian. It is well known that the difference between non-Hermitian Hamiltonian and Lindblad equation will become more significant as time evolves. It may not be a severe problem here, since the real evolution is only carried within a short time interval. But it will be of great interest to see how robust the dynamical topological invariant can be if one works in the more accurate Lindblad equation framework.

Reply: Thanks for the great question. The Lindblad equation provides a more accurate and complete description of the dynamics of open quantum systems, including population decay, phase decoherence, irreversible processes, etc. Over longer timescales, the Lindblad formalism is essential to model accurately the complete dynamics of open quantum systems, as it accounts for quantum jump terms that the non-Hermitian Hamiltonian approach does not capture.

First, we completely agree with you about “the difference between non-Hermitian Hamiltonian and Lindblad equation becomes more significant as time evolves”. Recently, another team in our group has observed the difference in a dissipative quantum gas [arXiv:2411.02910]. They use temporal quantum correlation to confirm that quantum jump terms play a crucial role in long-term evolution of a dissipative system.

Second, the reviewer is also right that “it may not be a severe problem here, since the real evolution is only carried within a short time interval”. In this work, when neglecting the quantum jump terms, the dynamics of the Lindblad master equation are equivalent to those of the non-Hermitian Hamiltonian over short timescales. During this period, the population leaving the effective two-level subspace does not interact back with the system, as illustrated in Figure 2 below.

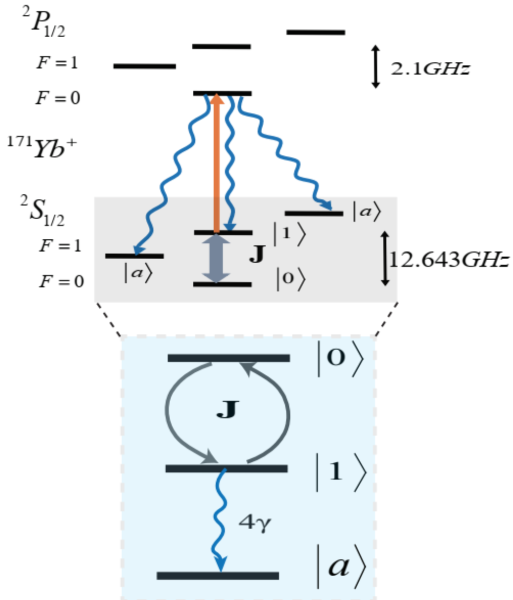


Figure 2. The experimental system of a passive PT-symmetric Hamiltonian.

Here we give a simple derivation process. The master equation can be described by

$$\frac{d\rho}{dt} = -i[H_C, \rho] + (L_1 \rho L_1^\dagger - \frac{1}{2}\{L_1^\dagger L_1, \rho\})$$

giving

$$\frac{d\rho}{dt} = \begin{pmatrix} iJ(\rho_{01}(t)-\rho_{10}(t)) & i(J\rho_{00}(t)+2i\gamma\rho_{01}(t)-J\rho_{11}(t)) & -iJ\rho_{1a}(t) \\ -2\gamma\rho_{10}(t)-iJ(\rho_{00}(t)-\rho_{11}(t)) & -iJ(\rho_{01}(t)-\rho_{10}(t))-4\gamma\rho_{11}(t) & -2\gamma\rho_{1a}(t)-iJ\rho_{0a}(t) \\ iJ\rho_{a1}(t) & -2\gamma\rho_{a1}(t)+iJ\rho_{a0}(t) & 4\gamma\rho_{aa}(t) \end{pmatrix} \quad \textbf{(a)}$$

When the system is initialized to the $|1\rangle$, the solution of Eq.(a) can be obtained with quantum jump term

$$\rho_{00}(t) = \frac{e^{-2t\gamma} J^2 \mathrm{Sin}[t\sqrt{J^2-\gamma^2}]^2}{J^2-\gamma^2}$$

$$\rho_{11}(t) = e^{-2t\gamma}\left(\mathrm{Cos}\left[t\sqrt{J^2-\gamma^2}\right] - \frac{\gamma \mathrm{Sin}[t\sqrt{J^2-\gamma^2}]}{\sqrt{J^2-\gamma^2}}\right)^2 \quad \textbf{(b)}$$

On the other hand, if we ignore the quantum jump term,

$$\frac{d\rho}{dt} = \begin{pmatrix} iJ(\rho_{01}(t) - \rho_{10}(t)) & i(J\rho_{00}(t) + 2i\gamma\rho_{01}(t) - J\rho_{11}(t)) & -iJ\rho_{1a}(t) \\ -2\gamma\rho_{10}(t) - iJ(\rho_{00}(t) - \rho_{11}(t)) & -iJ(\rho_{01}(t) - \rho_{10}(t)) - 4\gamma\rho_{11}(t) & -2\gamma\rho_{1a}(t) - iJ\rho_{0a}(t) \\ iJ\rho_{a1}(t) & -2\gamma\rho_{a1}(t) + iJ\rho_{a0}(t) & 0 \end{pmatrix} \quad \text{(c)}$$

It is obvious that discarding the quantum jump term does NOT affect the subspace in red color. In addition, the above conclusion can also be proved by comparing the evolution of with the effective Hamiltonian $H_{eff} = \begin{pmatrix} 0 & J \\ J & -2i\gamma \end{pmatrix}$ .The diagonal terms of the density matrix under $H_{eff}$ are given by:

$$\rho_{00}(t) = \left|\langle 0|\exp(-iH_{eff}(t))|1\rangle\right|^2 = \frac{e^{-2t\gamma}J^2\mathrm{Sin}[t\sqrt{J^2-\gamma^2}]^2}{J^2-\gamma^2}$$

$$\rho_{11}(t) = \left|\langle 1|\exp(-iH_{eff}(t))|1\rangle\right|^2 = e^{-2t\gamma}(\mathrm{Cos}\left[t\sqrt{J^2-\gamma^2}\right] - \frac{\gamma\mathrm{Sin}[t\sqrt{J^2-\gamma^2}]}{\sqrt{J^2-\gamma^2}})^2,$$ which

are the same as the solution given by Eq.(b), as shown in Figure3 below.

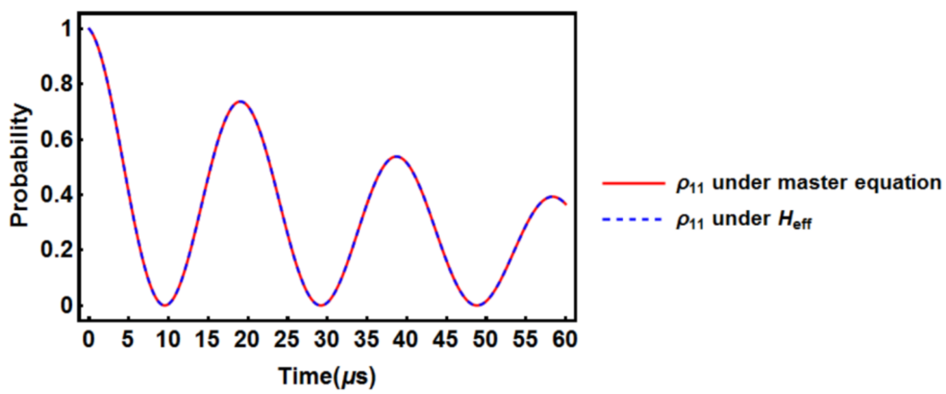


Figure 3. The comparison of the evolution of the master equation of the three-level and the evolution of the effective Hamiltonian. The simulation parameters are J = 0.16 MHz, γ/J = 0.05.

Third, we appreciate the reviewer's question about the robustness in Lindblad equation framework. The master equation, incorporating full quantum jump processes, describes the evolution of the density matrix under the Liouvillian superoperator. Since the Liouvillian can be represented as a non-Hermitian matrix acting on a vectorized density matrix, EPs can also emerge in the Liouvillian spectrum [AAPPS Bull. 34, 22 (2024)], leading to unique dynamics in the full quantum dynamics of the master equation.

Along the line of Liouvillian EPs, there are currently some theoretical and experimental works [Phys. Rev. Lett. 127. 140504, Phys. Rev. Lett. 128. 110402, Phys. Rev. A 104. L050405, Nat. Commun.13, 6225 (2022), Phys. Rev. Lett. 130. 110402, Phys. Rev. A. 108. 013302]. Among these, the encircling dynamics of LEPs was first experimentally studied in superconducting qubits [Phys. Rev. Lett. 127. 140504, Phys. Rev. Lett. 128. 110402], where chiral behaviors were reported to persist for adiabatic encircling.

Based on the above studies, we guess the topological invariant describe in our work (non-Hermitian Schrödinger equation) may be extended to the master equation. But we don’t have the conclusion because the impact of the quantum jump term on the dynamical topological invariants has yet to be explored. This requires further theoretical and experimental researches.

4. Words and labels in many figures are too small to be read.

Reply: We thank for the suggestion provided by the reviewer. In the revised manuscript, we have enlarged the words and labels of all the figures for reading.

5. There are no solid or dashed trajectories in Fig. 1(a) as mentioned in the caption.

Reply: We thank the reviewer for pointing this out. Solid or dashed trajectories should be in Fig. 1(b).

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Reviewer #2

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This is a review report of the manuscript "Dynamical topology of chiral and nonreciprocal state transfers in a non-Hermitian quantum system" by Pengfei Lu, Yang Liu et al. (COMMSPHYS-24-1114). The authors experimentally investigate a time-dependent, non-Hermitian trapped-ion system exhibiting exceptional points (EPs) on the eigenvalue Riemann sheets. By dynamically encircling the EPs, they observed intriguing topological chiral and nonreciprocal dynamics in both $\mathcal{PT}$-symmetric and $\mathcal{APT}$-symmetric Hamiltonians. Their results demonstrate that these dynamics are topologically robust against perturbations and are protected by a topological invariant-the dynamical vorticity-through the use of a parallel transported eigenbasis. Based on the comprehensive experimental results and well-formulated conclusions, I believe this work makes a significant contribution to the field of open quantum systems. It offers valuable insights into novel effects induced by EPs.

Reply: We appreciate the reviewer's positive comments.

Therefore, I recommend this paper for publication in Communications Physics, pending the authors' response to the following questions:

1) In Fig 3 (a), with a fixed $\tau_{crit}=11.8\mu s$, could the authors please explain why the contour lines of $\tau_{crit}/T$ (note: incorrectly labeled as $t_{crit}/T$ in the figure) appear irregular and jagged in the vertical direction (for a fixed encircling period $T$)? Are these contours derived from simulation or experimental results?

Reply: We thank the reviewer for pointing this out. The contours in Fig.3(a-b) of the manuscript are derived from numerical simulations. The irregular and jagged appearance of the contour lines in the vertical direction arises from irregularities in the original numerical simulation results, as shown in Figure 4 of the rebuttal letter. To better analyze the adiabaticity of dynamically encircling an EP across different periods, we utilized the parameter $\tau_{crit}/T$. When $\tau_{crit}/T \ll 1$, the adiabatic condition is well satisfied, resulting in a high-fidelity chiral (nonreciprocal) state transfer. Therefore, we performed post-processing on the original numerical data. To improve clarity without altering the data representation, we also displayed the graph with a reduced number of data points. The detailed processing steps are shown in Figure 4. As the reviewer pointing out "incorrectly labeled as $t_{crit}/T$ in the figure", we have revised the label into the right font in Figure 3 of the revised manuscript.

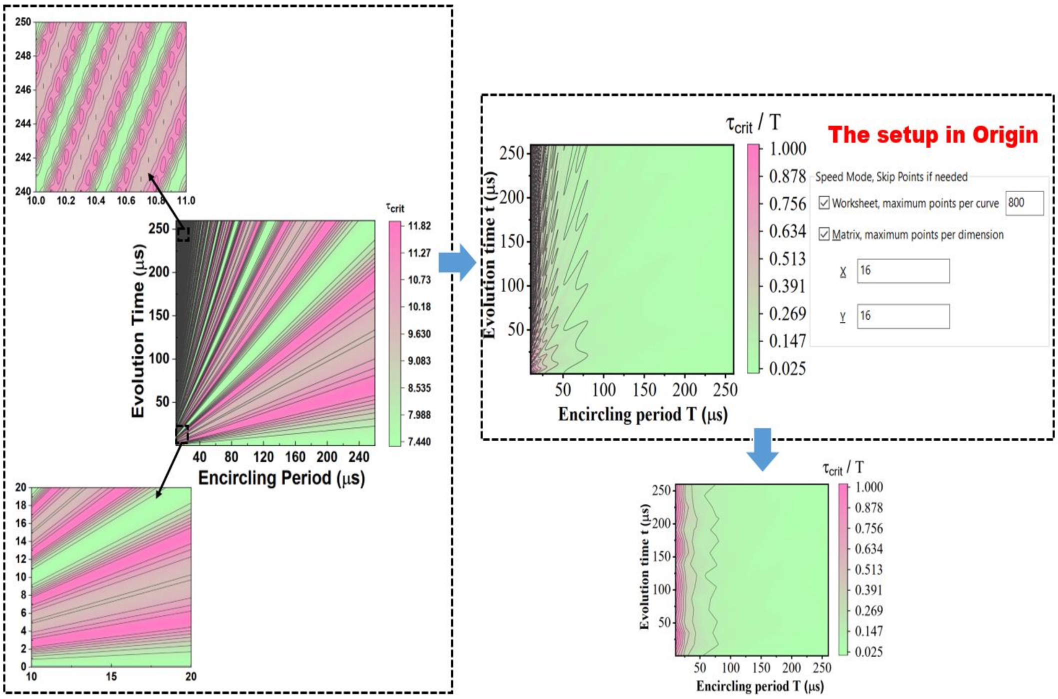


Figure 4. Schematic diagram of processing data

2) Regarding the instantaneous eigenstates $\ket{\alpha_A(t)}$ and $\ket{\beta_A(t)}$, do they represent the ideal adiabatic case where no DNAT occurs? For instance, in the trajectory 1 in Fig. 2, when the encircling angle reaches $2\pi$ at time $t$, does $\ket{\beta_A(t)}$ transition to $\ket{\alpha_A(0)}$? If this is the case, I would kindly suggest that the authors clarify this point when they first introduce these instantaneous eigenstates.

Reply: We thank for reviewer's questions and suggestions. The reviewer's guess is correct. The instantaneous eigenstates $|\alpha_A(t)\rangle$ and $|\beta_A(t)\rangle$ do represent the ideal adiabatic case where no DNAT occurs. They are non-orthogonal and represent the eigenstates of the time-dependent non-Hermitian Hamiltonian at each moment. When DNAT is absent (i.e., when the evolution is sufficiently slow), the system's state remains in its corresponding instantaneous eigenstate. However, if DNAT occurs, the system's evolution will no longer follow its instantaneous eigenstate.

For the trajectory 1 in Fig. 2, when the encircling angle reaches 2π at time t, the initial state $|\alpha_A(t=0)\rangle$ evolves to the final state $|\beta_A(t=250\mu s)\rangle$. According to the reviewer's suggestion, we have clarified these instantaneous eigenstates in the revised manuscript, which can be found in the second paragraph of page 4, as follows "Here, $|\alpha_A(t)\rangle$ and $|\beta_A(t)\rangle$ represent the eigenstates of the time-dependent non-Hermitian Hamiltonian at each moment."

3) As demonstrated in the abstract and introduction, in the absence of DNATs, the evolutionary state is expected to acquire a geometric phase of $\pi$ after encircling the EP twice, which is an indication of novel topology. However, I feel that the authors have not clearly demonstrated whether this geometric phase is reflected in their experimental results, or how it is manifested. Additionally, does the geometric phase of $\pi$ correspond to the half-integer dynamic vorticity $\mathcal{V_D}$ as defined by Eq. (4)? If so, could the authors please clarify why this is the case?

Reply: We thank for your questions. In the absence of DNATs, a parameter variation encircling an EP twice will acquire a geometric phase of $\pi$, as suggested by the theoretical analysis [Phys. Rev. A 72, 014104]. By using the **parametric** encirclement (without true dynamical evolution) to avoid the DNATs, several groups have demonstrated the $\pi$ phase change in the classical systems, e.g. in the dissipative microwave cavity [Phys. Rev. Lett. 86, 787], optical microcavity [Phys. Rev. Lett. 103, 134101, Nature 554, 526], and acoustic cavities [Phys. Rev. Lett. 127, 034301]. This phase defines an eigenvector winding number (VWN=1/2) after encircling an EP twice [Nat. Rev. Phys. 4, 745, Phys. Rev. Lett.127.034301].

In our work, because encircling the EP is a real **dynamical** process, so if we encircling the EP twice along the one direction, there must be at least one DNAT occurring (see trajectories 1 and 3 in Fig. 2 of the manuscript). This means that, compared to going back to the original state with a phase change of $\pi$ in parametric encirclement, it is impossible to return to the original state for dynamical encircling. The dynamical vorticity presented in this paper is a new concept and specifically associated with nonadiabatic transitions in dynamical evolution, while geometric phase of $\pi$ is mainly dedicated to the adiabatic process. So there is no direct relation between dynamic vorticity and the geometric phase, but the non-Hermiticity enables topological structures for both time-dependent properties (dynamics) and time-independent ones (eigenvectors), which is worth further study.

4) The authors demonstrated that the vorticity $\mathcal{V}_D$ equals to $\pm 1/2$ for trajectories 2 and 1, confirming the chiral state transfer. My question is: for the trajectories 5 and 6, where chiral symmetry is broken, what are the values of $\mathcal{V}_D$? Furthermore, as introduced in Ref [13], chiral transfer can still occur without encircling an EP, in which case, as demonstrated in this paper, $\mathcal{V}_D$ becomes 0. Does this mean that the topological invariant $\mathcal{V}_D$ fails in such a scenario?

Reply: We thank for reviewer's questions. First, we need to point out that the vorticity $\mathcal{V}_D$ is NOT used to confirm the chiral state transfer. This invariant is only used to characterize the robustness of the state transfer. For $\mathcal{V}_D = 1/2$, the state transfer is robustly protected by the topological invariant; however, the state transfer itself can exhibit either chiral or nonchiral behavior.

Second, for the trajectories 5 and 6, the vorticity $\mathcal{V}_D$ also equals to $\mp 1/2$. The positive or negative value of the vorticity depends on the orientation of encircling curve; it is negative if the curve travels around the EP in a clockwise direction. Although the chiral symmetry of trajectories 5 and 6 is broken, the state transfer is still robust because both trajectories protected by the topological invariant.

Third, the topological invariant does not fail in the scenario described in Ref. [13]. The dynamical vorticity $\mathcal{V}_D$ introduced in this work acts as a new topological invariant, defined as the winding number of the energy eigenvalues [Phys. Rev. Lett. 120. 146402]. This definition differs from conventional topological invariants, such as the Chern number, which typically characterize closed-loop behaviors in Hermitian systems. Here, $\mathcal{V}_D$ offers a new perspective for understanding topology in non-Hermitian systems with EPs.

To clarify, $\mathcal{V}_D$ is zero when the EP is not encircled. However, this zero-valued invariant does not imply the absence of a topological structure in the non-Hermitian system. Instead, it reflects a different kind of topology of encircling the EP or encircling near the EP. We use our dynamical vorticity to calculate the cases in Ref. [13] and find that, even when $\mathcal{V}_D = 0$, this invariant can indicate the robustness of chiral state transfer. This insight broadens the concept of topological invariants, demonstrating that a system's topological properties can be governed by distinct invariants based on the non-Hermitian Hamiltonian.

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Reviewer #3

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In this paper, authors discuss dynamical circling of exceptional points and experimentally observed chiral state transfers. The results sound reasonable, and the data are properly presented. However, the following points should be clarified before judging whether the manuscript warrants the publication.

Reply: We appreciate the reviewer's positive comment.

a) As discussed in "Chiral and nonreciprocal state transfers", dissipation-induced nonadiabatic transitions (DNATs) result in chiral state transfers. However, it is unclear what its mechanism is and when it occurs. This point should be clarified.

Reply: Thanks for the question. In non-Hermitian systems, energy eigenvalues are generally complex. Here, the real parts of the eigenvalues represent energy levels, while the imaginary parts correspond to dissipation (loss) or amplification (gain) in the system. Unlike Hermitian systems, where adiabatic evolution is typically maintained if parameters change slowly enough, the presence of complex eigenvalues introduces exotic behaviors in non-Hermitian systems.

In this work, the imaginary parts of the eigenvalues, arising from dissipation effects, lead to what are known as nonadiabatic transitions that convert a system toward a preferred state; that is, an evolutionary state in a loss sheet will jump to gain sheet, even when the parameters evolve slowly. On the other hand, in a system with a gain, a slight increase in the gain state will accumulate and amplify over time, ultimately causing the system to be dominated by the gain. Therefore, an adiabatic evolution invariably converts a system to its gain state. An analytic explanation of the chiral state transfer can also be found in Ref [Phys. Rev. Lett. 118. 093002]. It should be noted that such transitions are underlain by the geometry of the complex eigenvalue manifold (see Fig.2(c) and Fig.2(e) in the manuscript).

This nonadiabatic transition process is also direction-dependent when encircling an EP. For example, if the system encircles the EP in a clockwise direction (trajectory 1), the evolving state remains in the gain sheet throughout, leading it to evolve into a different eigenstate in the absence of nonadiabatic transitions. Conversely, in a counterclockwise path (trajectory 2), where the state does not consistently stay in the gain sheet, the presence of nonadiabatic transitions allows the system to evolve back into its original state. This direction-dependence of the encircling path yields different final states, which defines the chiral state transfer. Thus, we conclude that: 1) dissipation-induced nonadiabatic transitions (DNATs) result in chiral state transfers, 2) the state transfers have chiral behavior if the encircling processes experience different numbers of DNATs. This can be found in the third paragraph on page 4 of the original manuscript.

b) The authors claim that Eq. (4) is their new result. However, it is essentially the same as the vorticity defined in Ref. 20 [PRL 120 146402 (2018)]. The authors should explain the difference.

Reply: Thanks for your comment. The appearance of Eq. (4) in our paper looks the same as the definition in [PRL 120 146402 (2018)], but the physical meaning of the eigenvalues in the formula are distinctively different. The fundamental difference lies in that the vorticity defined in Ref. 20 [PRL 120 146402 (2018)] is used for stationary non-Hermitian Hamiltonian, while we proposed the dynamical vorticity for time-dependent non-Hermitian Hamiltonian.

To explain the difference, we have added explanations of the differences between them in "Discussion" section of the original article, which can be found in the pages 11 and 12, as follows

"Based on the above results, we address this question by providing a new topological invariant that we dub as "dynamic vorticity" for characterizing topological protections involving DNATs. Previously, the vorticity, also known as the spectral or eigenvalue winding number[20,21,31,32], is defined as $\nu_{mn}(\Gamma) = -1/2\pi \oint_{\Gamma} \nabla_k \arg[\mathrm{E}_m(k) - \mathrm{E}_n(k)] \cdot dk$. Here eigenvalues $\mathrm{E}_m(k)$ and $\mathrm{E}_n(k)$ are varied with momentum $k$ ($k \in [0,2\pi]$) in the Brillouin zone and the indices m and n represent different band structures. Noted that, in this scenario, although eigenvalues parametrically change along the close loop $\Gamma$ in the momentum space, there are no dynamics or nonadiabatic transitions driven by the time-dependent Hamiltonian. Under this condition, the vorticity presents the topological structure of a static Hamiltonian. In our case, we deal with the time-dependent Hamiltonian, which generates real time dynamics, including nonadiabatic transition.

To effectively explain the robustness of the chiral and nonreciprocal dynamics that we observed experimentally. We find that the vorticity concept can be extended to the time-dependent Hamiltonian. To include the dynamics, we adopt the parallel transported eigenbasis[33], giving

$$\widetilde{H}_{eff}(t) = \begin{pmatrix} -i\gamma - \lambda(t) & f(t) \\ -f(t) & -i\gamma + \lambda(t) \end{pmatrix}, \qquad (3)$$

the nonadiabatic transition process is reflected in the off-diagonal term of Eq. (3) since $f(t) = \frac{J(\dot{\Delta}/2+i\dot{\gamma})-\dot{J}(\Delta/2+i\gamma)}{2i\lambda^2}$ includes the time derivatives of both the coupling $J$ and the detuing $\Delta$, as seen from Eq. (15) (Details in Materials and methods). Then, the dynamic vorticity is given by

$$\nu_D = -1/2\pi \oint_{\Gamma} \nabla_\theta \arg\left[\widetilde{\mathrm{E}}_+(\theta) - \widetilde{\mathrm{E}}_-(\theta)\right] \cdot d\theta, \qquad (4)$$

where $\widetilde{\mathrm{E}}_\pm(\theta = \omega t)$ are dynamically varied with encircling angle $\theta$ ($\theta \in [0,2\pi]$) in the parameter space. The dynamic vorticity associates with the energy dispersion of $\widetilde{H}_{eff}(t)$, which characterizes the topology of dynamics of encircling the EP. The validity of $\nu_D$ lies in the fact that, in the parallel transported basis, nonadiabatic transitions will not cause the deviations of the instantaneous state from the Riemann surfaces. Consequently, the system's evolution consistently remains in the eigenstate of $\widetilde{H}_{eff}(t)$. Therefore, $\nu_D$ is a universal topological invariant near the EP not only for adiabatic dynamics, but also for nonadiabatic ones. It is also noted that the definition and form of Eq. (4) apply to both PT-symmetric and passive PT-symmetric Hamiltonians, since introducing an extra diagonal term does not alter the basic topology of the dynamics."

In conclusion, the vorticity defined in Ref. 20 [PRL 120 146402 (2018)] is targeted for stationary non-Hermitian Hamiltonian, while the dynamical vorticity proposed in our paper is suitable for time-dependent non-Hermitian Hamiltonian.

c) SNAT before "Methods" is not defined.

Reply: We have already defined SNAT in "Classification of nonadiabatic transitions" section of the original article. The specific place where this definition appears is in the third paragraph of page 9.

## Responses to Reviewers' Questions and Comments

In the following, please find a detailed reply to the reviewers' comments. The review reports are reproduced below in blue, and our point-by-point responses are in black.

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Reviewer #2

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Thank you for your detailed and thorough responses. I believe the changes and explanations you provided have effectively addressed my initial concerns.

However, based on your reply to my question (4), where you stated that ' the vorticity $\mathcal{V}_D$ is not used to confirm the chiral state transfer' and that ' even when $\mathcal{V}_D = 0$, this invariant can indicate the robustness of the chiral state transfer,' it appears that the chiral state transfers can occur both when $\mathcal{V}_D = 0$ and when $\mathcal{V}_D \neq 0$. Therefore, the statement on the bottom of Page 12, '... confirming chiral and nonreciprocal state transfers are topologically protected.' may not be entirely appropriate in this context. I suggest further clarifying this point in the main article to avoid potential misunderstandings for readers.

Overall, with this clarification, I believe the manuscript will be suitable for publication.

Reply: We thank for the reviewer's suggestion. In the revised manuscript, we have updated the statement on the bottom of Page 12, as follows

"In our experiments shown in Fig. 2, we observed that $\mathcal{V}_D = \pm 1/2$, i.e., $-1/2$ for trajectory 1 and 1/2 for trajectory 2. We further find that $\mathcal{V}_D = 0$ for processes that do not encircle an EP, a result consistent with the state transfers investigated in Refs [10,13]. Therefore, both the half-integer and zero-valued invariants characterize the robustness of the state transfers. Thus, we conclude that $\mathcal{V}_D = \pm n/2$, where n represents the number of EPs being encircled and the sign ± depends on the orientation of the encircling curve."