

Supplemental Material for: Dynamics of a helical vortex ring interacting with a vortex line

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(Dated: April 4, 2025)

Supplementary Note 1: HELICITY IN IDEAL FLUIDS

In ideal fluid dynamics, a perfect vortex ring possesses energy and linear impulse. However, the perturbation of a Kelvin wave enables the vortex ring to acquire angular impulse and helicity [1]. In ideal fluid dynamics, helicity is an essential conserved quantity that characterizes the linking and knotting of vortices. It is defined as

$$\mathcal{H} = \sum_{i \neq j} \Gamma_i \Gamma_j Lk_{ij} + \sum_i \Gamma_i^2 (Tw_i + Wr_i) \quad (\text{S1})$$

where Lk_{ij} is the linking number between vortex lines i and j . Γ , Tw_i , and Wr_i are the circulation, twist, and writhe of vortex line i , respectively [2]. The helical vortex ring carries the writhe, while the embedding of vortex lines introduces the twist to the vortex ring [1, 3].

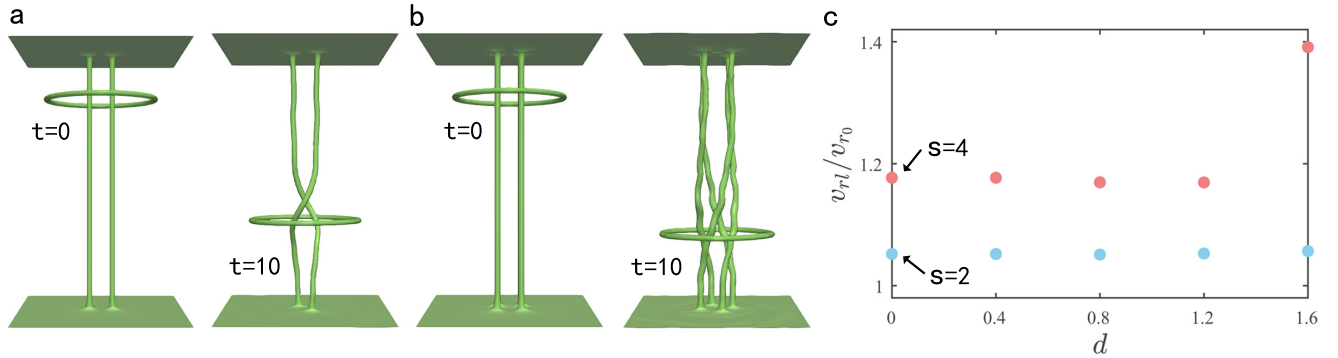


FIG. S1. The dynamics of a vortex ring with radius $R_r = 3$ in the case of surrounding separated vortex lines sitting symmetrically about the axial of the vortex ring. (a) Two separated vortex lines with a separation distance of $d = 1.2$. (b) Four separated vortex lines, with the distance $d = 1.2$ between any two vortex lines in the x and y directions. (c) The relative speed of the vortex ring v_{rl}/v_{r0} as a function of the separation distance of the vortex lines d .

Supplementary Note 2: SPLITTING OF VORTEX LINES

In comparison to the results of vortex lines with high topological charge, we run simulations to explore the influence of multiple initially separated vortex lines on the dynamics of the vortex ring. Figs. S1(a) and (b) depict scenarios with two and four initially separated vortex lines, respectively, demonstrating that the vortex ring maintains its overall shape during motion. Fig. S1(c) presents a comparison of the relative speed of the vortex ring at various initial separation distances d of the vortex lines. Our findings suggest that for small separation distances ($d \leq 1.2$ in our dataset), the impact on the vortex ring's speed aligns with that of highly charged vortex lines. Conversely, at larger separation distances ($d = 1.6$ in our dataset), the impact of separated vortex lines on the vortex ring's speed is more pronounced than that of highly charged vortex lines. However, overall, the separation still leads to an increase in the vortex ring's speed if there is no connection between the vortex lines and the vortex ring.

Furthermore, as depicted in Fig. S2(a), we conducted computations involving scenarios in which the vortex line doesn't align with the axis of the vortex ring, thereby introducing asymmetry to the system. Introducing a vortex line with a unit topological charge at an offset position from the vortex ring axis revealed a corresponding increase in

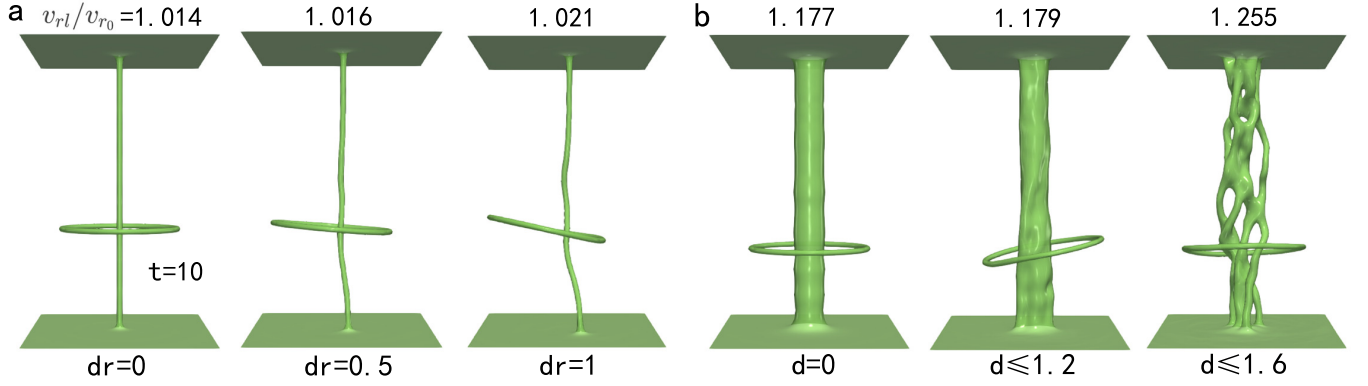


FIG. S2. Dynamics of a vortex ring with $R_r = 3$. (a) The distances of the vortex lines from the vortex ring axis are $d_r = 0, 0.5$, and 1 , respectively. (b) Vortex lines are randomly placed, with the distances between the vortex lines in the x and y directions being less than or equal to $0, 1.2$, and 1.6 , respectively.

the speed of the vortex ring proportional to the offset distance. This adjustment resulted in observable inclinations of the vortex ring and distortions in the vortex line. Fig. S2(b) illustrates the impact of introducing four randomly placed vortex lines within the region enclosed by the vortex ring, demonstrating a similar trend of increased vortex ring speed with the distance d between the lines ($d = 0$ is the case of a multi-charged vortex line). For the largest offset distances up to 1.2 , the speed enhancement is relatively minor compared to that induced by centrally positioned multi-charged vortex lines. Notably, increasing the maximum offset distance between the lines to 1.6 led to a more significant enhancement in the vortex ring speed. These findings align with those displayed in Fig. S1, suggesting a consistent and robust increase in vortex ring speed. Furthermore, this acceleration is influenced by the distribution of topological charges within the region encompassed by the vortex ring.

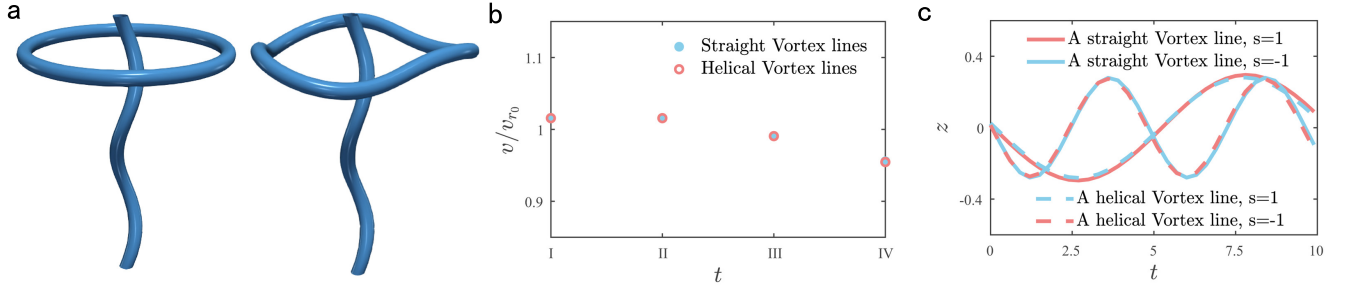


FIG. S3. Dynamics of vortex rings with helical vortex lines. (a) The density isosurfaces of a helical vortex line are inserted into a perfect vortex ring and a helical vortex ring, respectively. (b) Translational speed of vortex rings under different conditions. Case I: a perfect vortex ring with a $s = +1$ vortex line. Case II: a perfect vortex ring with a $s = -1$ vortex line. Case III: a helical vortex ring with a $s = +1$ vortex line. Case IV: a helical vortex ring with a $s = -1$ vortex line. Squares and circles represent the straight vortex line and the helical vortex line, respectively. (c) The time evolution of the position of the two-dimensional point vortex (see Fig. 3(f) in the main text) in the z -direction. The red (blue) solid line represents the case of a helical vortex ring with a $s = +1$ ($s = -1$) straight vortex line. The blue (red) dashed line represents the case of a helical vortex ring with a $s = +1$ ($s = -1$) helical vortex line.

Supplementary Note 3: HELICAL VORTEX LINES

Straight vortex lines have pronounced impacts on both the translational and rotational dynamics of helical vortex rings, raising the question: what influence do helical vortex lines have on the dynamics of these vortex rings? In Fig. S3(a), we show two scenarios: one with a helical vortex line within a perfect vortex ring and the other with a helical vortex line within a helical vortex ring. Fig. S3(b) illustrates eight scenarios involving helical vortex lines and vortex rings, revealing that helical vortex lines have strikingly similar effects on the translational speed of vortex rings as straight vortex lines. Likewise, the rotational dynamics of a helical vortex ring are impacted by a helical vortex line similarly to a straight vortex line, as depicted in Fig. S3(c).

Supplementary Note 4: INITIAL CONFIGURATION OF HELICAL VORTEX RINGS

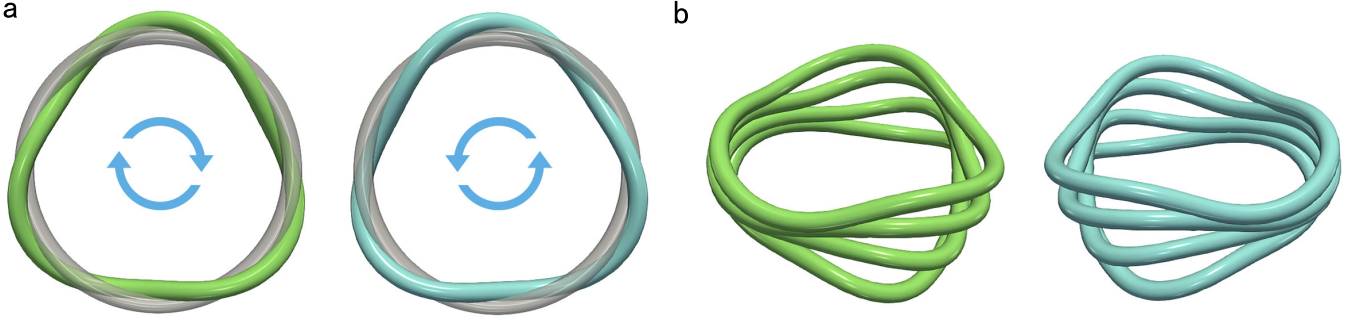


FIG. S4. Initial construction of helical vortex rings. (a) Initial shapes of helical vortex rings rotating clockwise and counter-clockwise are depicted, with the blue arrow indicating the direction of rotation. (b) Rotational dynamics of the vortex rings, displaying density isosurfaces from $t = 0$ to $t = 3$.

The rotation direction of a helical vortex ring is influenced by its initial configuration. In Figs. S4 (a) and (b), we depict two different vortex ring configurations, demonstrating how the initial shape dictates the rotation direction of the helical vortex ring. In the main text, we present the initial phase for a clockwise-rotating vortex ring, while the initial wave function for a counterclockwise-rotating vortex ring is given by

$$\begin{aligned} \psi_{3D}(x, y, z) = & \psi_{2Dr} \{r - R_r - A \sin[n\theta(x, y)], z - z_0 + A \cos[n\theta(x, y)]\} \times \\ & \psi_{2Dr}^* \{r + R_r - A \sin[n\theta(-x, -y)], z - z_0 + A \cos[n\theta(-x, -y)]\}, \end{aligned} \quad (\text{S2})$$

where $\psi_{2Dr}(r, z) = \sqrt{N_0} \exp[i\theta(r, z)]$, $\theta(x, y) = \text{atan2}(y, x)$, N_0 is the total number of atoms, z_0 is the initial position of the center of mass of the helical ring along the z -axis, A and n are the amplitude and the wave number of Kelvin waves, respectively.

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