

Supplementary Information:

Generalization of non-Hermitian spectral topology to hyperbolic lattices

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Section I. Geometry of hyperbolic lattices

In this section, we provide a brief introduction to the geometry of hyperbolic lattices (HBLs), including their general features and the method of constructing them on the Poincaré disk.

The HBL is typically classified as $\{p, q\}$ type, signifying that the plane is tessellated by regular p -gons, with each lattice site having a coordination number of q . Take an infinite $\{p, q\}$ HBL into account, which can be divided into unit cells, each containing a given number of sites, organized within a hyperbolic Bravais lattice $\{p_B, q_B\}$ ¹. The translation group associated with this Bravais lattice is a Fuchsian group Γ , which is isomorphic to the fundamental group of a genus- g ($g > 1$) surface and can be described by the presentation $\Gamma = \langle \gamma_1, \gamma_2, \dots, \gamma_{2g} \mid X_g \rangle$. Here, X_g represents a single relator with each generator γ_i ($i = 1, \dots, 2g$) and its inverse emerging once. Such characteristics unique to HBLs are intensively explored in the Refs.^{1,2}

Although HBLs intrinsically reside in a high-dimensional space, in practice, they can be directly visualized by using the Poincaré disk if the unit cells and Bravais lattices are known. One can start by putting down the Poincaré-disk coordinates of all sites in the central unit cell and establishing the PSU(1,1) matrix representation of each generator γ_i . Then, the HBL can be constructed by applying all the products of representations to the central unit cell¹.

For instance, considering the $\{p, q\} = \{8, 8\}$ HBL we employ in the main text, which displays itself as a $\{p_B, q_B\} = \{8, 8\}$ Bravais lattice with a single site in each unit cell. The coordinate of the site in the central unit cell is $w_0 = 0 + 0i$. In this case, $g = 2$ and the corresponding representation can be expressed as $\gamma_i = R((i-1)\alpha_B)\gamma_1 R(-(i-1)\alpha_B)$ with $i = 1, 2, 3, 4$. Here, $\gamma_1 = \frac{1}{\sqrt{(1-\sigma^2)}} \begin{pmatrix} 1 & \sigma \\ \sigma & 1 \end{pmatrix}$ with $\sigma = \sqrt{\frac{\cos(\alpha_B) + \cos(\beta_B)}{1 + \cos(\beta_B)}}$ and $\alpha_B = \beta_B = 2\pi/8$ is the first generator, and R is the rotational matrix written by $R(\theta) = \begin{pmatrix} \exp(i\theta/2) & \sigma \\ \sigma & \exp(i\theta/2) \end{pmatrix}$. The action of all γ_i on the complex coordinate z is determined by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$. Consequently, the HBL on the Poincaré disk can be constructed by applying the products of these four generators to the central unit cell. As illustrated in Fig. 1b, we depict the central unit cell along with its eight adjacent cells, obtained through translations $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_1^{-1}, \gamma_2^{-1}, \gamma_3^{-1}, \gamma_4^{-1}\}$.

Section II. The single-band nonreciprocal model

In this section, we show the equality $\sigma_U^{(1)} = \sigma_U^{(2)}$, provide the OBC spectral distributions, and give details of the spatial localization of the NHSE for the single-band nonreciprocal model.

A. Analyses of $\sigma_U^{(1)}$ and $\sigma_U^{(2)}$

For the single-band nonreciprocal model on the $\{8, 8\}$ lattice discussed in the main text, we find that $\sigma_U^{(1)} = \sigma_U^{(2)}$ (Fig. 1e, f). In this section, we offer details on the determination of $\sigma_U^{(1)}$ and $\sigma_U^{(2)}$, demonstrating their equality.

We first investigate the Hamiltonian of the primitive cell:

$$H^{(1)}(\boldsymbol{\beta}^{(1)}) = 2\beta_1 + \beta_1^{-1} + \beta_2 + \beta_2^{-1} + \beta_3 + \beta_3^{-1} + \beta_4 + \beta_4^{-1}, \quad (\text{S1})$$

with $\beta_j^{(1)} = e^{ik_j + \mu_j}$ ($j = 1, \dots, 4$). We can clearly see that the non-Hermiticity comes from nonreciprocal nearest-neighbor coupling along the γ_1 direction. Notice that $H^{(1)}(\boldsymbol{\beta})$ can be decomposed into $H_1^{(1)}(\beta_1) + H_2^{(1)}(\beta_2) + H_3^{(1)}(\beta_3) + H_4^{(1)}(\beta_4)$, with only the first term exhibiting the nonreciprocal property of the model. In consequence, the determination of $\sigma_U^{(1)}$ should be attributed to $\text{Sp}_\ell^{(1)}$ with only $|\beta_1| \neq 1$. For a quantitative solution, we redefine $\beta_j^{(1)} = r_j e^{ik_j}$ with $r_j = |\beta_j^{(1)}| = e^{\mu_j}$ ($j = 1, \dots, 4$), and then the real and imaginary parts of eigenvalue E can be expressed as:

$$\text{Re}(E) = \left(2r_1 + \frac{1}{r_1}\right)\cos(k_1) + \left(r_2 + \frac{1}{r_2}\right)\cos(k_2) + \left(r_3 + \frac{1}{r_3}\right)\cos(k_3) + \left(r_4 + \frac{1}{r_4}\right)\cos(k_4), \quad (\text{S2a})$$

$$\text{Im}(E) = \left(2r_1 - \frac{1}{r_1}\right)\sin(k_1) + \left(r_2 - \frac{1}{r_2}\right)\sin(k_2) + \left(r_3 - \frac{1}{r_3}\right)\sin(k_3) + \left(r_4 - \frac{1}{r_4}\right)\sin(k_4). \quad (\text{S2b})$$

To obtain the maximal $\text{Re}(E)$ for each $\text{Sp}_\ell^{(1)}$ (denoted as E_r), we need to set $k_j = 0$ such that $\cos(k_j) = 1$ in Eq. (S2a):

$$E_r = 2r_1 + \frac{1}{r_1} + r_2 + \frac{1}{r_2} + r_3 + \frac{1}{r_3} + r_4 + \frac{1}{r_4}. \quad (\text{S3})$$

Next, to identify the range of the real part of $\sigma_U^{(1)}$, we are supposed to find the $\text{Sp}_\ell^{(1)}$ with minimal E_r . Thus, we compute the partial derivatives of E_r with respect to r_j :

$$\frac{\partial E_r}{\partial r_1} = 2 - \frac{1}{r_1^2}, \quad (\text{S4a})$$

$$\frac{\partial E_r}{\partial r_2} = 1 - \frac{1}{r_2^2}, \quad (\text{S4b})$$

$$\frac{\partial E_r}{\partial r_3} = 1 - \frac{1}{r_3^2}, \quad (\text{S4c})$$

$$\frac{\partial E_r}{\partial r_4} = 1 - \frac{1}{r_4^2}, \quad (\text{S4d})$$

and the positive solution ($r_j \geq 0$) of $\frac{\partial E_r}{\partial r_j} = 0$ is $r_1 = \frac{1}{\sqrt{2}}$ and $r_2 = r_3 = r_4 = 1$. This satisfies $\frac{\partial^2 E_r}{\partial r_j^2} >$

0, resulting in the minimal $E_r = 6 + 2\sqrt{2}$. From Eq. (S2b), it follows that the above solution

ensures $\text{Im}(E) = 0$ for all $\{k_j\}_{j=1}^4$, which is the minimal range of imaginary parts of $\text{Sp}_\ell^{(1)}$.

Therefore, by setting $|\beta_1| = \frac{1}{\sqrt{2}}$ and $|\beta_2| = |\beta_3| = |\beta_4| = 1$, the $\text{Sp}_\ell^{(1)}$ with minimal spectral range

is achieved, lying on the real axis and corresponding to the $\sigma_U^{(1)}$ (Fig. 1e).

Next, we delve into $H^{(2)}(\boldsymbol{\beta}^{(2)})$ with $\beta_j^{(2)} = e^{ik_j + \mu_j}$ ($j = 1, \dots, 6$) for the 2-supercell, written as:

$$H^{(2)}(\boldsymbol{\beta}^{(2)}) = \begin{pmatrix} 0 & F[\boldsymbol{\beta}^{(2)}] \\ G[\boldsymbol{\beta}^{(2)}] & 0 \end{pmatrix}. \quad (\text{S5})$$

Here, $F[\boldsymbol{\beta}^{(2)}] = 1 + \beta_1 + \beta_2 + \beta_1\beta_3^{-1} + \beta_2\beta_4^{-1} + \beta_1\beta_3^{-1}\beta_5 + \beta_2\beta_4^{-1}\beta_6 + 2\beta_1\beta_2\beta_5\beta_6$ and

$G[\boldsymbol{\beta}^{(2)}] = 2 + \beta_1^{-1} + \beta_2^{-1} + \beta_1^{-1}\beta_3 + \beta_2^{-1}\beta_4 + \beta_1^{-1}\beta_3\beta_5^{-1} + \beta_2^{-1}\beta_4\beta_6^{-1} + \beta_1^{-1}\beta_2^{-1}\beta_5^{-1}\beta_6^{-1}$.

Although $H^{(2)}(\boldsymbol{\beta}^{(2)})$ is much more complex than $H^{(1)}(\boldsymbol{\beta}^{(1)})$, we are capable of determining $\sigma_U^{(2)}$

by employing a procedure similar to that used for $\sigma_U^{(1)}$. Without the need to present tedious

formulas, we outline the procedure below. We again redefine $\beta_j^{(2)} = r_j e^{ik_j}$ with $r_j = |\beta_j^{(2)}| = e^{\mu_j}$

($j = 1, \dots, 6$), and find $\{r_j\}_{j=1}^6$ that minimizes the $\text{Sp}_\ell^{(2)}$. Mathematically, to simplify the calculation,

we first consider $\text{Re}(E^2)$ and $\text{Im}(E^2)$ because $E^2 = F[\boldsymbol{\beta}^{(2)}]G[\boldsymbol{\beta}^{(2)}]$. Clearly, $\text{Re}(E^2)$ [$\text{Im}(E^2)$]

can then be also expressed as a sum of multiple cosine (sine) terms. As a result, by setting k_j to zero in $\text{Re}(E^2)$, taking the partial derivative with respect to r_j , and equating it to zero, we can obtain the solution $\{r_j\}_{j=1}^6$ that minimizes the range of $\text{Re}(E^2)$, which is $|r_1| = |r_2| =$ and $|r_3| = |r_4| = |r_5| = |r_6| = 1$. It can be verified that this solution simultaneously ensures $\text{Im}(E^2) = 0$. Furthermore, this solution minimizes the ranges of both $\text{Re}(E)$ and $\text{Im}(E)$, thereby minimizing $\text{Sp}_\ell^{(2)}$. Accordingly, as we set $|\beta_1| = |\beta_2| = \frac{1}{\sqrt{2}}$ and $|\beta_3| = |\beta_4| = |\beta_5| = |\beta_6| = 1$, $\text{Sp}_\ell^{(2)}$ is minimized and corresponds to $\sigma_U^{(2)}$ that lies on the real axis (Fig. 1f). Since the maximal real part of $\sigma_U^{(2)}$ is also $6 + 2\sqrt{2}$, we conclude that $\sigma_U^{(1)} = \sigma_U^{(2)}$.

B. OBC spectra

We have shown that $\sigma_U^{(1)} = \sigma_U^{(2)}$ for the single-band nonreciprocal model, with both spectra lying on the real axis (Fig. 2). Here, we present several OBC spectra for this model in Fig. S1, all lying on the real axis, which further demonstrates that our method for determining the OBC spectral range of nonreciprocal models is indeed feasible.

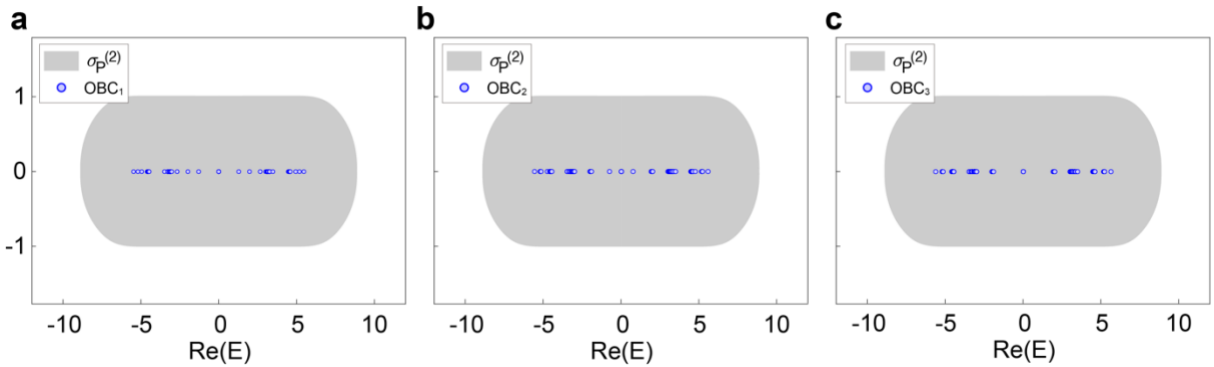


Fig. S1. OBC spectra of single-band nonreciprocal lattices. For the three structures with site numbers 1142, 2168, and 3193 in **a**, **b**, and **c**, respectively, the OBC spectra (blue circles) lie on the real axis. Regions $\sigma_p^{(2)}$ are plotted in grey.

C. Spatial localization of the NHSE

To better visualize the spatial localization of the NHSE in Fig. 2, we have plotted the spectral sums of the eigenstates (over the same interval as in the KL divergence plot of Fig. 2b) across the site index, as shown in Fig. S2a. The spectral sum is defined as $\frac{1}{N} \sum_n |\psi_n^R|^2$, where ψ_n^R is the n -th normalized right eigenvector of the Hamiltonian and N is the number of eigenvectors. Here, we select an arbitrary position along the γ_4 -axis and present the data in the remaining three-dimensional space. The spatial localization along the γ_1^{-1} direction is clearly observed, indicating the presence of the NHSE.

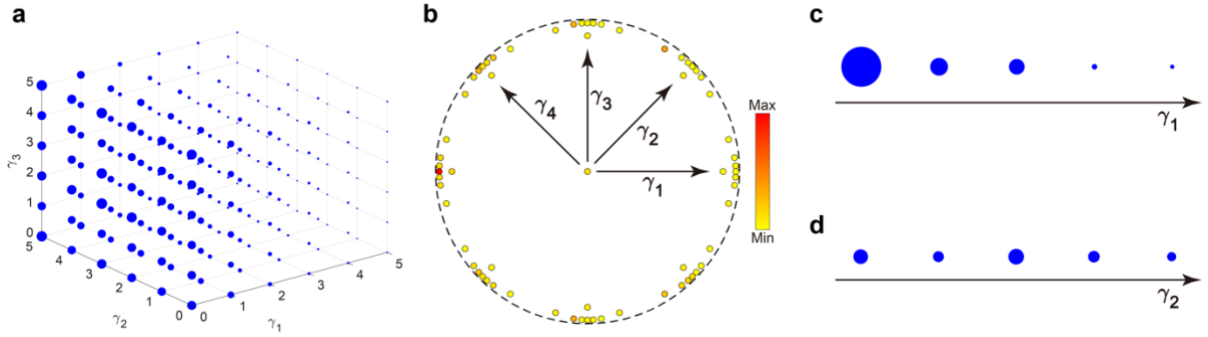


Fig. S2. **a** Spectral sum of eigenstates for the HDEL (over the same interval as in the KL divergence plot of Fig. 2b) plotted versus site index. Here, we select an arbitrary position along the γ_4 -axis and display the data in the remaining three-dimensional subspace. **b** Spectral sum of eigenstates for a hyperbolic $\{8,8\}$ lattice under OBC; the lattice contains 65 sites. **c**, **d** Spectral sums of eigenstates along **c** γ_1 and **d** γ_2 directions for the hyperbolic lattice in **b**. In **a**, **c**, and **d**, marker sizes are proportional to the magnitudes of spectral sums.

Moreover, to further illustrate eigenstate accumulation due to the NHSE, we investigate a specific hyperbolic $\{8,8\}$ lattice on the Poincaré disk (Fig. S2b). By introducing nonreciprocal couplings in the same direction as those for the HDEL in Fig. 2 but with larger magnitudes to enhance visualization, the distribution of eigenstates becomes nonuniform. Specifically, we compute the spectral sum on sites along the γ_1 direction (Fig. S2c) and clearly observe a tendency

for the eigenstates to localize along the γ_1^{-1} direction, providing direct evidence of the NHSE in the hyperbolic lattice. In contrast, the spectral sum along the γ_2 direction does not exhibit a directional tendency (Fig. S2d), further confirming the nonreciprocal NHSE.

We also present the spectral sums of the left eigenvectors for both the higher-dimensional Euclidean lattice (Fig. S2a) and the finite $\{8,8\}$ lattice with OBC (Fig. S2b) in Fig. S3. Compared to Fig. S2, one can clearly see that the left eigenvectors mainly accumulate on the side opposite to that of the right eigenvectors. This opposing localization is a hallmark of the nonreciprocal NHSE.

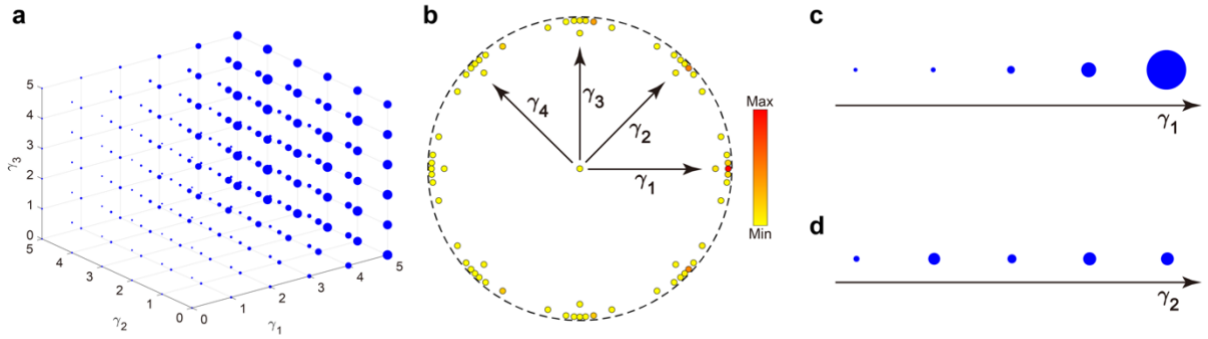


Fig. S3. Spectral sums of the left eigenvectors for both **a** the higher-dimensional Euclidean lattice and **b-d** the finite $\{8,8\}$ lattice with OBC discussed in Fig. S2.

Section III. Non-Hermitian non-Abelian semimetals

In this section, we present the details for obtaining the $U(2)$ spectrum (Fig. 4b) and the convergence property of $\sigma_U^{(n)}$ (Fig. 4c).

A. $U(2)$ spectra by $U(2)$ -HBT

According to Fig. 4a, when we choose $m = 3 + 0.3i$, the 2-supercell should exhibit a semimetal phase, which can be further corroborated by exploring the $U(d)$ -HBT³. In this subsection, we provide details for the data presented in Fig. 4b, utilizing the approach introduced in the Methods. Specifically, we randomly choose 30 $SU(2)$ matrices $\{U_j\}_{j=1}^4$ and 10^6 $U(1)$ factors $\{e^{ik_j}\}_{j=1}^4$, clearly indicating the gapless property of the $U(2)$ spectrum. It is worth noting that the

SU(2) matrices $\{U_j\}_{j=1}^4$ and the U(1) factors $\{e^{ik_j}\}_{j=1}^4$ lie in the 6-dimensional and 4-dimensional spaces, respectively, meaning that our random sampling is possible to be non-homogeneous. However, this does not influence our conclusion, as the gapless feature is unequivocal. In Fig. S4a, b, we illustrate the U(2) spectrum in Fig. 4b and the U(1) spectrum, respectively. The gapless (gapped) U(2) [U(1)] spectrum unambiguously signifies that the semimetal behavior inherently comes from higher-dimensional irreducible representations.

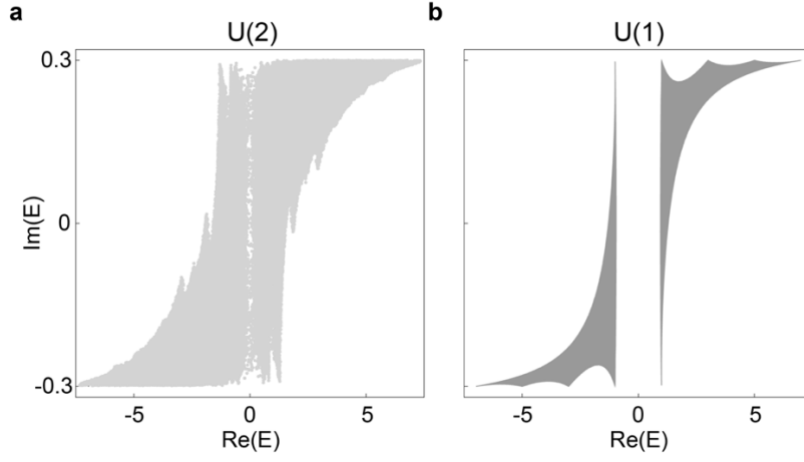


Fig. S4. **a** U(2) spectrum obtained by randomly selecting 30 SU(2) matrices and 10^6 U(1) factors. **b** U(1) spectrum. Here, $m = 3 + 0.3i$ for both cases.

B. Convergence of $\sigma_U^{(n)}$

In Fig. 4c, we depict $\sigma_U^{(1)}$, $\sigma_U^{(4)}$, and two OBC spectra when $m = 5 + 0.5i$. The degenerate states under OBCs lying in $\sigma_U^{(4)} - \sigma_U^{(1)}$ indicate that the existence of non-Abelian states. In this subsection, we show the convergence of $\sigma_U^{(n)}$ versus n , which confirms that the non-Abelian states appear in the thermodynamic limit.

For each n , $\det[H^{(n)}(\boldsymbol{\beta}^{(n)}) - E] = \det[H^{(n)}(\boldsymbol{\beta}^{-1(n)}) - E] = 0$ is true, such that the Ronkin function always satisfies $R_f(\boldsymbol{\mu}) = R_f(-\boldsymbol{\mu})$, guaranteeing that $\sigma_U^{(n)} = \sigma_P^{(n)4,5}$. We plot $\sigma_U^{(n)}$ for $n = 1$ and $n = 4$ in Figs. S5a and S5b, respectively, which clearly show the change in the

spectral region as non-Abelian states are taken into account. To display the convergence of the spectra in the thermodynamic limit, we focus on three particular points [$\text{Re}(E) = -4, -5, -6$], highlighted by arrows in Fig. S5a, b, and show their imaginary parts for incremental n ($= 1, 2, 4, 8, 16, 32$) in Fig. S5c. The convergent property of the spectra, as well as the emergence of non-Abelian states in the thermodynamic limit, has been clearly seen.

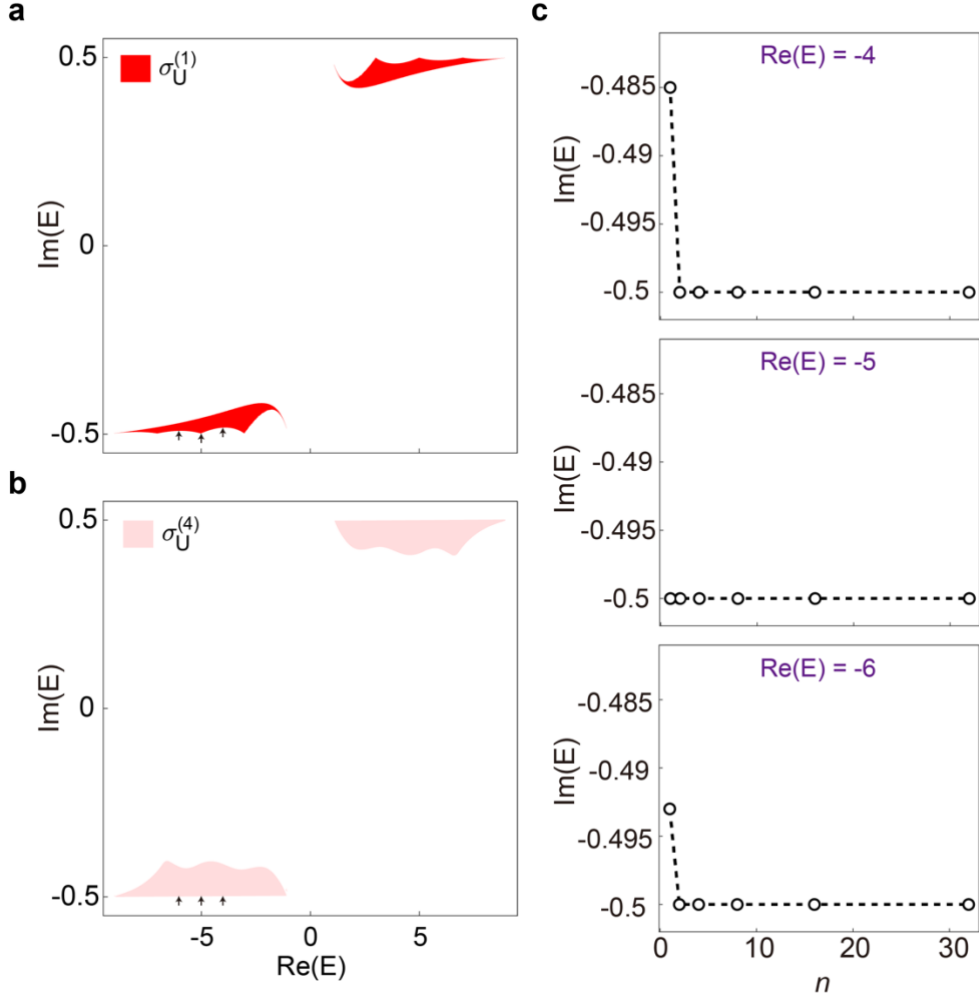


Fig S5. $\sigma_U^{(n)}$ for **a** $n = 1$ and **b** $n = 4$. **c** The upper, middle, and bottom panels show the minimal imaginary parts of eigenvalues for eigenstates with real parts -4 , -5 , and -6 , respectively. Here, $m = 5 + 0.5i$ for each n .

Section IV. Sequence of the supercells

In this section, we provide details of the supercell sequence and discuss its convergence to the thermodynamic limit.

A. Details of the sequence

In our work, we use the supercell sequence and adopt the corresponding sequence of normal subgroups introduced in Ref. 6. Specifically, supercells are defined via quotient groups G of the underlying space group (more precisely, its proper subgroup) by the translation group $\Gamma^{7,8}$. For the $\{8, 8\}$ lattice, the proper subgroup of the space group is the triangle group

$$\Delta^+(2,8,8) = \langle x, y, z \mid x^2, y^8, z^8, xyz \rangle,$$

which is generated by rotations x , y , and z around the three vertices of a hyperbolic triangle by angles $2\pi/2$, $2\pi/8$, and $2\pi/8$, respectively.

To achieve convergence to the thermodynamic limit, the sequence of translation groups $\Gamma^{(m)} \triangleleft \Delta^+(2,8,8)$ needs to obey the normal subgroup relations $\Gamma^{(1)} \triangleright \Gamma^{(2)} \triangleright \dots \triangleright \Gamma^{(m)} \triangleright \dots$. The minimal unit cell (primitive cell, depicted in Fig. 1b) is obtained as the smallest quotient of $\Delta^+(2,8,8)$ by one of its normal subgroups $\Gamma^{(1)}$

$$G^{(1)} = \langle x, y, z \mid x^2, y^8, z^8, xyz, xzy, y^3 z^{-1} \rangle.$$

$G^{(1)}$ is isomorphic to the quotient group $\Delta^+/\Gamma^{(1)}$, with $\Gamma^{(1)} = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \mid \gamma_1 \gamma_2^{-1} \gamma_3 \gamma_4^{-1} \gamma_1^{-1} \gamma_2 \gamma_3^{-1} \gamma_4 \rangle$. Using GAP^{9,10}, this normal sequence of translation subgroups $\Gamma^{(m)}$ is found by the appropriate quotient groups $G^{(m)} \cong \Delta^+/\Gamma^{(m)}$, chosen so that each supercell preserves maximal symmetry, and then these quotient groups can be presented as:

$$\begin{aligned} G^{(1)} &= \langle x, y, z \mid x^2, y^8, z^8, xyz, xzy, y^3 z^{-1} \rangle, \\ G^{(2)} &= \langle x, y, z \mid x^2, y^8, z^8, xyz, xzy \rangle, \\ G^{(3)} &= \langle x, y, z \mid x^2, y^8, z^8, xyz, xy^{-2} z^{-1} y, xzy^{-1} z^{-2} \rangle, \\ G^{(4)} &= \langle x, y, z \mid x^2, y^8, z^8, xyz, xy^{-2} z^{-1} y \rangle, \\ G^{(5)} &= \langle x, y, z \mid x^2, y^8, z^8, xyz, xzy^{-2} z^{-2} y, (yz^{-1} y^2)^2 \rangle, \\ G^{(6)} &= \langle x, y, z \mid x^2, y^8, z^8, xyz, xzy^{-2} z^{-2} y \rangle. \end{aligned}$$

These correspond to symmetric supercells; note that the superscript (m) indicates the position of the subgroup in the coherent sequence, not the supercell index. For example, $G^{(2)}$ and $G^{(3)}$

correspond to the 2-supercell and 4-supercell, respectively. Since all supercells preserve eightfold rotational symmetry and central mirror symmetries, this method provides a systematic way to construct sequences of PBC clusters in a symmetric manner.

B. Discussion on the convergence to the thermodynamic limit

Reference 7 demonstrates that the supercell sequences can be extended to satisfy the assumption $\cap_{m \geq 1} \Gamma^{(m)} = \{1\}$ using two approaches, which is required by Ref. 6. First, they note that while the supercell method does not yield a unique sequence of normal subgroups—allowing many different sequences for the same lattice and primitive cell—these sequences converge to the same limit. Second, they show that the results from the supercell method align with those obtained via the continued-fraction method in real space, further supporting the convergence of the supercell method to the thermodynamic limit.

Our work strictly uses the sequences of normal subgroups, which have been shown to satisfy the assumption by Ref. 6. Moreover, the convergence of $\sigma_{\text{U}}^{(n)}$ is numerically demonstrated in Section III.B of the Supplementary Information, confirming the feasibility of our method.

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