Supplementary Information for: Memory loss is contagious in open quantum systems

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Supplementary Note 1: The derivation of the BR LS approach

The Master equation describing the evolution of the density matrix $\rho_{s+b}(t)$ that encompasses the degrees of freedom of both a lossy system and an external bath is expressed as follows:

$$\frac{d\rho_{s+b}(t)}{dt} = -\frac{i}{\hbar}[\hat{H}_s + \hat{H}_b + \hat{H}_{sb}, \rho_{s+b}(t)] + \sum_A \gamma_A \hat{L}_{\hat{A}} \rho_{s+b}(t). \tag{S1}$$

Here, $\hat{H}_s, \hat{H}_b$ represent the Hamiltonians of the system and bath, respectively. The interaction system-bath term, $\hat{H}_{sb}$, is given by $\hat{V}_s \otimes \int_0^\infty d\omega \sqrt{J_b(\omega)}(b_\omega + b_\omega^\dagger)$, where $\hat{V}_s$ is the system coupling operator, $J_b(\omega)$ is the bath's spectral density and b_ω ($b_\omega^\dagger$) is the bath bosonic annihilation (creation) operators. Additionally, $\hat{L}_{\hat{A}}\rho(t)$ is the Lindblad superoperator, given by $\hat{A}\rho(t)\hat{A}^\dagger - \frac{1}{2}\{\hat{A}^\dagger\hat{A}, \rho(t)\}$, for the system jump operator $\hat{A}$ with the rate γ_A [1].

First, we transform Eq. (S1) into the system-bath interaction picture that enables treating the bath perturbatively through the second-order Born approximation. To achieve this, we omit the terms $\hat{A}\rho(t)\hat{A}^\dagger$ in the Lindblad superoperator, responsible for incoherent transitions. This results in the equation: $\frac{d\rho_{s+b}(t)}{dt} = -\frac{i}{\hbar}[\hat{\mathcal{H}}_{s+b}\rho_{s+b}(t) - \rho_{s+b}(t)\hat{\mathcal{H}}_{s+b}^\dagger]$, where $\hat{\mathcal{H}}_{s+b} = \hat{\mathcal{H}}_s + \hat{H}_b + \hat{H}_{sb}$ and $\hat{\mathcal{H}}_s = \hat{H}_s - \frac{i}{2}\sum_A \gamma_A \hat{A}^\dagger\hat{A}$ is the effective non-Hermitian (NH) Hamiltonian describing the system dynamics [2]. The transformation to the interaction picture is then achieved using the non-unitary operator $U(t) = e^{i(\hat{\mathcal{H}}_s + \hat{H}_b)t/\hbar}$, yielding a master equation for the density matrix in the interaction picture $\rho_{s+b}^I(t) = U(t)\rho_{s+b}(t)U^\dagger(t)$:

$$\frac{d\rho_{s+b}^I(t)}{dt} = -\frac{i}{\hbar}\left(\hat{H}_{sb}^I(t)\rho_{s+b}^I(t) - \rho_{s+b}^I(t)\hat{H}_{sb}^{I\dagger}(t)\right) \tag{S2}$$

where $\hat{H}_{sb}^I(t) = e^{i(\hat{\mathcal{H}}_s + \hat{H}_b)t/\hbar}\hat{H}_{sb}e^{-i(\hat{\mathcal{H}}_s + \hat{H}_b)t/\hbar}$ and $\hat{H}_{sb}$ is Hermitian. Integrating Eq. (S2) and substituting the resulting $\rho_{s+b}^I(t)$ into it, we obtain:

$$\begin{aligned}\frac{d\rho_{s+b}^I(t)}{dt} = &-\frac{i}{\hbar}\left(\hat{H}_{sb}^I(t)\rho_{s+b}^I(0) - \rho_{s+b}^I(0)\hat{H}_{sb}^{I\dagger}(t)\right) \\ &- \frac{1}{\hbar^2}\int_0^t dt'\Big(\hat{H}_{sb}^I(t)\hat{H}_{sb}^I(t')\rho_{s+b}^I(t') - \hat{H}_{sb}^I(t)\rho_{s+b}^I(t')\hat{H}_{sb}^{I\dagger}(t') \\ &- \hat{H}_{sb}^I(t')\rho_{s+b}^I(t')\hat{H}_{sb}^{I\dagger}(t) + \rho_{s+b}^I(t')\hat{H}_{sb}^{I\dagger}(t')\hat{H}_{sb}^{I\dagger}(t)\Big).\end{aligned} \tag{S3}$$

Next, we assume that the interaction is perturbative, such that $\rho_{s+b}(t) \approx \rho_s(t) \otimes \rho_b$ and ρ_b is the time-independent and thermally populated density matrix of the bath. Tracing the bath's degrees of freedom results in a master equation for the system density matrix in the interaction picture $\rho_s^I(t) = \mathrm{Tr}_B[\rho_{s+b}^I(t)] = e^{i\hat{\mathcal{H}}_s t/\hbar}\rho_s(t)e^{-i\hat{\mathcal{H}}_s^\dagger t/\hbar}$, given by:

$$\begin{aligned}\frac{d\rho_s^I(t)}{dt} = &-\frac{1}{\hbar^2}\int_0^t dt'\Big[C(t-t')\Big(\hat{V}_s^I(t)\hat{V}_s^I(t')\rho_s^I(t') \\ &-\hat{V}_s^I(t')\rho_s^I(t')\hat{V}_s^{I\dagger}(t)\Big) + C^*(t-t')\Big(\rho_s^I(t')\hat{V}_s^{I\dagger}(t')\hat{V}_s^{I\dagger}(t) \\ &-\hat{V}_s^I(t)\rho_s^I(t')\hat{V}_s^{I\dagger}(t')\Big)\Big].\end{aligned} \tag{S4}$$

Here, $\hat{V}_s^I(t) = e^{i\hat{\mathcal{H}}_s t/\hbar}\hat{V}_s e^{-i\hat{\mathcal{H}}_s t/\hbar}$ and $C(\tau) = \int_0^\infty d\omega J_b(\omega)\{(n(\omega)+1)e^{-i\omega\tau} + n(\omega)e^{i\omega\tau}\}$ is the autocorrelation function of a thermal bosonic bath with the spectral density $J_b(\omega)$ where $n(\omega)$ is the Bose occupation number.

The BR LS approach is based on the assumption that the system's loss of coherence is responsible for the Markovianity of the interaction with the bath. Since the interaction picture used above does not capture the system loss, we cannot use it when assuming the Markovian approximation. Therefore, we apply on Eq. (S4) the inverse transformation to retrieve the Lindblad terms:

$$\frac{d\rho_s(t)}{dt} = -\frac{i}{\hbar}[\hat{H}_s, \rho_s(t)] + \sum_A \gamma_A \hat{L}_{\hat{A}}\rho_s(t) \; - \frac{1}{\hbar^2}\int_0^t d\tau\Big[C(\tau)\times \Big(\hat{V}_s e^{-i\hat{\mathcal{H}}_s\tau}\hat{V}_s\rho_s(t-\tau)e^{i\hat{\mathcal{H}}_s^\dagger\tau} - e^{-i\hat{\mathcal{H}}_s\tau}\hat{V}_s\rho_s(t-\tau)e^{i\hat{\mathcal{H}}_s^\dagger\tau}\hat{V}_s\Big) + C^*(\tau)\Big(e^{-i\hat{\mathcal{H}}_s\tau}\rho_s(t-\tau)\hat{V}_s e^{i\hat{\mathcal{H}}_s^\dagger\tau}\hat{V}_s - \hat{V}_s e^{-i\hat{\mathcal{H}}_s\tau}\rho_s(t-\tau)\hat{V}_s e^{i\hat{\mathcal{H}}_s^\dagger\tau}\Big)\Big]. \tag{S5}$$

Then, to remove the operators in the exponents, we use the identity operators described by the right and left eigenfunctions of $\hat{\mathcal{H}}_s$ and $\hat{\mathcal{H}}_s^\dagger$, $\hat{I} = \sum_a |a)(a|$ and $\hat{I} = \sum_a |a^*)(a^*|$, where $\hat{\mathcal{H}}_s|a) = \hbar(\omega_a - i\frac{\Gamma_a}{2})|a)$, $(a|\hat{\mathcal{H}}_s = \hbar(\omega_a - i\frac{\Gamma_a}{2})(a|$, $\hat{\mathcal{H}}_s^\dagger|a^*) = \hbar(\omega_a + i\frac{\Gamma_a}{2})|a^*)$, and $(a^*|\hat{\mathcal{H}}_s^\dagger = \hbar(\omega_a + i\frac{\Gamma_a}{2})(a^*|$. Note that we use the notation $|\dots)$, $(\dots|$ rather than $|\dots\rangle$, $\langle\dots|$ to describe the right and left eigenstates of the NH Hamiltonian [3]. The states fulfill $(a^*| = |a)^\dagger = \langle a|$ (where $|a) = |a\rangle$) and $(a|b) = \delta_{ab}$, but not $(a^*|b) = \delta_{ab}$. In addition, we define $(a|\hat{V}_s|b) = \tilde{V}_{ab}$, which implies $(a^*|\hat{V}_s|b^*) = \tilde{V}_{ab}^*$ since $\hat{V}_s$ is a real and Hermitian operator. Consequently,

$$\frac{d\rho_s(t)}{dt} = -\frac{i}{\hbar}[\hat{H}_s, \rho_s(t)] + \sum_A \gamma_A \hat{L}_{\hat{A}}\rho_s(t) - \frac{1}{\hbar^2}\sum_{p,q,r,s}\int_0^t d\tau\Big[C(\tau)\times \Big(\tilde{V}_{pq}e^{-i(\omega_q - \frac{i}{2}\Gamma_q)\tau}\tilde{V}_{qr}|p)(r|\rho_s(t-\tau)|s^*)(s^*|e^{i(\omega_s + \frac{i}{2}\Gamma_s)\tau} - e^{-i(\omega_q - \frac{i}{2}\Gamma_q)\tau}\tilde{V}_{qp}|q)(p|\rho_s(t-\tau)|s^*)(r^*|e^{i(\omega_s + \frac{i}{2}\Gamma_s)\tau}\tilde{V}_{sr}^*\Big) + C^*(\tau)\times \Big(e^{-i(\omega_s - \frac{i}{2}\Gamma_s)\tau}|s)(s|\rho_s(t-\tau)|r^*)(p^*|\tilde{V}_{rq}^* e^{i(\omega_q + \frac{i}{2}\Gamma_q)\tau}\tilde{V}_{qp}^* - \tilde{V}_{rs}e^{-i(\omega_s - \frac{i}{2}\Gamma_s)\tau}|r)(s|\rho_s(t-\tau)|p^*)(q^*|\tilde{V}_{pq}^* e^{i(\omega_q + \frac{i}{2}\Gamma_q)\tau}\Big)\Big]. \tag{S6}$$

One may notice that each term inside the integral in Eq. (S6) includes the decaying exponent $e^{-\frac{1}{2}(\Gamma_q+\Gamma_s)\tau}$, which diminishes when $\tau \to \infty$. This decaying exponent encodes the impact of the system loss on its interaction with the bath and facilitates the Markovian assumption within the derivation of the BR LS approach for $\tau > \tau_{qs} = \frac{2}{\Gamma_q + \Gamma_s}$. We use the coherent frame of the eigenenergies' difference to avoid artifacts originating from the loss of the memory of the internal dynamics of the system and assume $(i|\rho_s(t-\tau)|j^*) \approx (i|\rho_s(t)|j^*)e^{i(\omega_i - \omega_j)\tau}$. Moreover, we expand the upper limit of the integral to ∞. Subsequently, Eq. (S6) becomes Markovian and reads as:

$$\frac{d\rho_s(t)}{dt} = -\frac{i}{\hbar}[\hat{H}_s, \rho_s(t)] + \sum_A \gamma_A \hat{L}_{\hat{A}}\rho_s(t) - \frac{1}{\hbar^2}\int_0^\infty d\tau\times \sum_{p,q,r,s}\Big[C(\tau)\Big(\tilde{V}_{pq}\tilde{V}_{qr}|p)(r|\rho_s(t)|s^*)(s^*|e^{i(\omega_r - \omega_q)\tau}e^{-\frac{1}{2}(\Gamma_s+\Gamma_q)\tau} - \tilde{V}_{qp}\tilde{V}_{sr}^*|q)(p|\rho_s(t)|s^*)(r^*|e^{i(\omega_p - \omega_q)\tau}e^{-\frac{1}{2}(\Gamma_s+\Gamma_q)\tau}\Big) + C^*(\tau)\Big(\tilde{V}_{rq}^*\tilde{V}_{qp}^*|s)(s|\rho_s(t)|r^*)(p^*|e^{-i(\omega_r - \omega_q)\tau}e^{-\frac{1}{2}(\Gamma_s+\Gamma_q)\tau} - \tilde{V}_{rs}\tilde{V}_{pq}^*|r)(s|\rho_s(t)|p^*)(q^*|^{-i(\omega_r - \omega_q)\tau}e^{-\frac{1}{2}(\Gamma_s+\Gamma_q)\tau}\Big)\Big]. \tag{S7}$$

Finally, we define $F_{jsq} = \int_0^\infty d\tau C(\tau) e^{-i(\omega_q-\omega_j)\tau} e^{-\frac{1}{2}(\Gamma_q+\Gamma_s)\tau}$, and substitute $C(\tau)$ in it:

$$
\begin{aligned}
F_{jsq} = \int_0^\infty \int_0^\infty d\tau d\omega e^{-\frac{1}{2}(\Gamma_q+\Gamma_s)\tau} J_b(\omega) \times \\
\left[(n(\omega)+1) e^{-i(\omega_q-\omega_j+\omega)\tau} + n(\omega) e^{-i(\omega_q-\omega_j-\omega)\tau} \right] = \\
\int_0^\infty d\omega J_b(\omega) \left[(n(\omega)+1) \frac{\frac{1}{2}(\Gamma_q+\Gamma_s) - i(\omega_q-\omega_j+\omega)}{(\omega_q-\omega_j+\omega)^2 + \frac{1}{4}(\Gamma_q+\Gamma_s)^2} \right. \\
\left. + n(\omega) \frac{\frac{1}{2}(\Gamma_q+\Gamma_s) - i(\omega_q-\omega_j-\omega)}{(\omega_q-\omega_j-\omega)^2 + \frac{1}{4}(\Gamma_q+\Gamma_s)^2} \right].
\end{aligned}
\tag{S8}
$$

This simplifies Eq. (S7), leading to the final Master equation within the BR LS approach, given by:

$$
\begin{aligned}
\frac{d\rho_s(t)}{dt} = -\frac{i}{\hbar}[\hat{H}_s, \rho_s(t)] + \sum_A \gamma_A \hat{L}_{\hat{A}} \rho_s(t) \\
-\frac{1}{\hbar^2} \sum_{p,q,r,s} \Big[F_{rsq} \tilde{V}_{pq} \tilde{V}_{qr} |p)(r|\rho_s(t)|s^*)(s^*| \\
- F_{psq} \tilde{V}_{qp} \tilde{V}^*_{sr} |q)(p|\rho_s(t)|s^*)(r^*| \\
+ F^*_{rsq} \tilde{V}^*_{rq} \tilde{V}^*_{qp} |s)(s|\rho_s(t)|r^*)(p^*| \\
- F^*_{psq} \tilde{V}_{rs} \tilde{V}^*_{pq} |r)(s|\rho_s(t)|p^*)(q^*| \Big],
\end{aligned}
\tag{S9}
$$

and can be rewritten by left multiplication with $(a|$ and right multiplication with $|b^*)$:

$$
\begin{aligned}
\frac{d(a|\rho_s(t)|b^*)}{dt} = -i\left(\omega_a - \omega_b - i\frac{\Gamma_a+\Gamma_b}{2}\right)(a|\rho_s(t)|b^*) \\
+ \sum_{c,d} \left(\sum_A \gamma_A (a|\hat{A}|c)(d^*|\hat{A}^\dagger|b^*) - \frac{1}{\hbar^2} R_{abcd} \right) (c|\rho_s(t)|d^*),
\end{aligned}
\tag{S10}
$$

where $R_{abcd} = \delta_{bd} \sum_q F_{cdq} \tilde{V}_{aq} \tilde{V}_{qc} + \delta_{ac} \sum_q F^*_{dcq} \tilde{V}^*_{dq} \tilde{V}^*_{qb} - (F_{cda} + F^*_{dcb}) \tilde{V}_{ac} \tilde{V}^*_{db}$ is the BR LS tensor used to solve the system dynamics.

Supplementary Note 2: Extracting an effective spectral density from BR LS

The BR LS results appearing in the figures in the main text are obtained without any further approximations to Eq. (S10). However, to extract an effective spectral density from BR LS, we utilize the secular approximation where $(n|\rho_s(t)|m^*) = \delta_{nm}(n|\rho_s(t)|n^*)$. Under this approximation, we substitute $a = b = f$ and $c = d = i$ in Eq. (S10). Moreover, we assume that the imaginary part of $\tilde{V}_{if}$ is negligible. This gives

$$
\begin{aligned}
\frac{d(f|\rho_s(t)|f^*)}{dt} = -\left(\Gamma_f + \sum_q (1-\delta_{qf}) \frac{2\,\mathrm{Re}[F_{ffq}]|\tilde{V}_{fq}|^2}{\hbar^2} \right) \times \\
(f|\rho_s(t)|f^*) + \sum_i \left(\sum_A \gamma_A (f|A|i)(i^*|A^\dagger|f^*) \right. \\
\left. + (1-\delta_{if}) \frac{2\,\mathrm{Re}[F_{iif}]|\tilde{V}_{if}|^2}{\hbar^2} \right) \times (i|\rho_s(t)|i^*).
\end{aligned}
\tag{S11}
$$

We notice that the rate constant for the population transfer from state i to state f is given by

$$
\begin{aligned}
K^{\mathrm{BRLS}}_{i\to f} = \frac{2\,\mathrm{Re}[F_{iif}]|\tilde{V}_{if}|^2}{\hbar} = 2\pi \int_0^\infty d\omega J_b(\omega) \times \\
\Big[(n(\omega)+1) T_{i\to f}(\omega) + n(\omega) T_{i\to f}(-\omega) \Big] |\tilde{V}_{if}|^2,
\end{aligned}
\tag{S12}
$$

where $T_{i\to f}(\omega) = \frac{1}{\pi\hbar}\frac{\frac{1}{2}(\Gamma_i+\Gamma_f)}{(\omega_f-\omega_i+\omega)^2+\frac{1}{4}(\Gamma_f+\Gamma_i)^2}$ is the spectrum of the transition $i \to f$, broadened due to the losses. By comparing $K^{\mathrm{BRLS}}_{i\to f}$ with the constant rate obtained by BR, $K^{\mathrm{BR}}_{i\to f} = 2\pi J(\omega_i-\omega_f)|\tilde{V}_{if}|^2$ [4] when $n(\omega)=0$, we extract an effective spectral density for each transition $i \to f$, given in Eq. (8) in the main text.

Supplementary Note 3: Isolation of the dynamics induced by the non-Markovian bath using the NH formalism

In the example shown in Fig. 2 of the main text, the Markovian bath induces only irreversible decay. This allows us to isolate the influence of the non-Markovian bath on the system's dynamics: any population transfer between the eigenstates of the non-Hermitian system Hamiltonian $\hat{\mathcal{H}}_s$ within the same excitation manifold can be attributed solely to the non-Markovian bath.

To demonstrate this, we simulate the full Lindblad equation with the same system parameters and eigenstates as in Fig. 2, but with the coupling to the non-Markovian bath set to zero, i.e., $g_b = 0$ and $J_b(\omega) = 0$. Fig. S1(a) shows the resulting population of the lower polariton (LP), which, as introduced in the main text, is an eigenstate of $\hat{\mathcal{H}}_s$ defined in Eq. (5) within the first excitation manifold. Initially, only the upper polariton, another eigenstate of $\hat{\mathcal{H}}_s$ in the same manifold, is populated. As seen, no population transfer occurs in this case and the LP population remains zero over time. This confirms that Fig. 2 isolates the effect of the non-Markovian bath.

For comparison, Fig. S1(b) shows the population of the lower eigenstate of the Hermitian system Hamiltonian (the real part of Eq. (5) in the main text) in the first excitation manifold, using the same parameters as in (a), but when the initial state is the corresponding upper eigenstate. Here, in contrast to Fig. S1(a), population transfer is observed even in the absence of the non-Markovian bath, demonstrating the advantage of studying population dynamics in the eigenbasis of $\hat{\mathcal{H}}_s$, where transfer directly reflects the influence of the non-Markovian environment.

We emphasize that the ability to isolate the contribution of the non-Markovian bath in this specific case relies on the decay-only structure of the Markovian bath. The validity of the BR LS approach does not depend on this simplification and remains applicable more broadly, including cases where the Markovian bath induces dephasing, as discussed in the Supplementary Note 7.

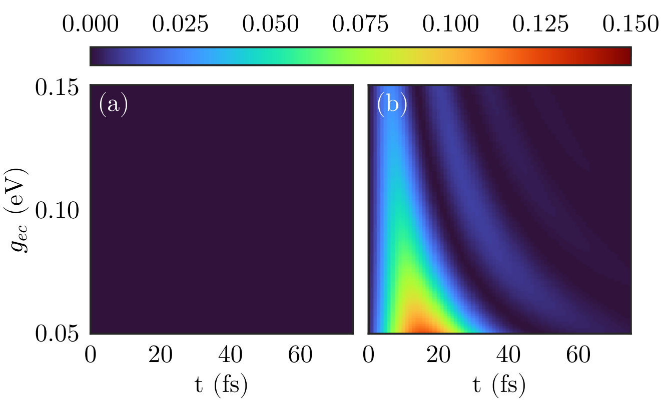

FIG. S1: The population transfer between the eigenstates of (a) the NH effective Hamiltonian, (b) the Hermitian Hamiltonian, when $g_b = 0$, i.e., no interaction with the non-Markovian bath.

Supplementary Note 4: Trace preservation of BR LS

Similarly to the standard BR approach, the BR LS conserves the trace of the density matrix. This can be shown analytically from Eq. (S5): taking the trace of both sides yields a vanishing time derivative of the trace, indicating that the evolution preserves normalization. To confirm this numerically, we tracked the trace of the density matrix at each time step for all values of g_{ec} shown in Fig. 2(b) in the main text. The deviation from unity remained on the order of 10^{-14} throughout the simulations, as shown in Fig. S2, confirming the preservation of the trace by BR LS.

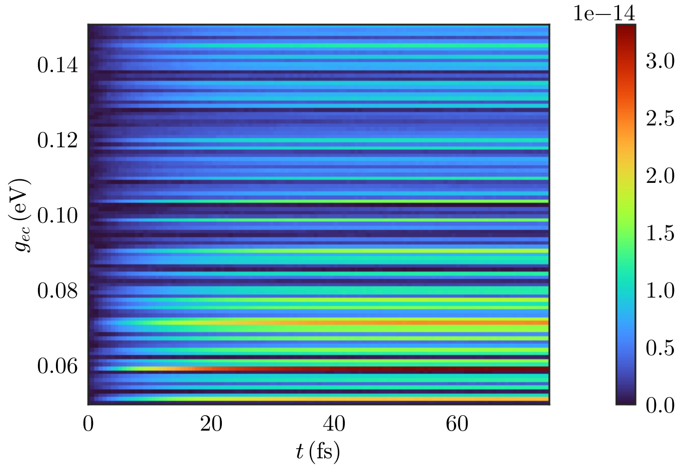

FIG. S2: Deviation of the trace of the density matrix from unity as a function of time, for various values of g_{ec}, corresponding to the BR LS simulations shown in Fig. 2(b) in the main text.

Supplementary Note 5: The performance of BR LS in obtaining the elements of the system's density matrix

In Fig. S3(a), we present the absolute difference between the exact results presented in Fig. 2(a) in the main text and the BR LS results presented in Fig. 2(b) in the main text. For comparison, in Fig. S3(b), we present the absolute difference between the exact and the BR (Fig. 2(c) in the main text) results. As can be seen, while both BR LS and BR present a deviation from the exact results due to approximations made in their derivations, BR LS reproduces much better agreement than BR, particularly for times longer than the lifetime of the states, i.e., $t > \tau_p = \frac{1}{\Gamma_p} \approx 13$ fs, marked by the vertical dashed white line.

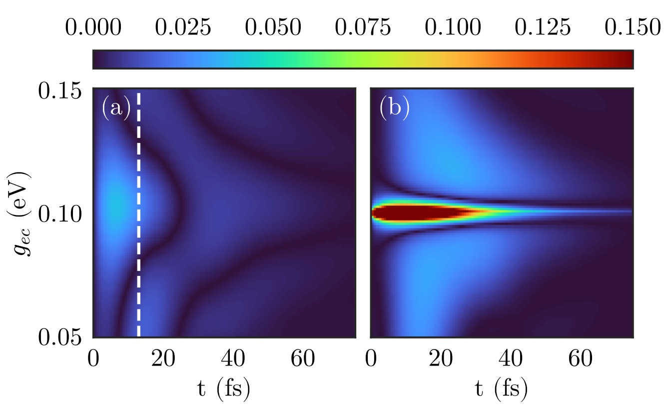

FIG. S3: The deviation of (a) BR LS and (b) BR results from the exact results presented in Fig. 2 in the main text. The dashed white line denotes the lifetime of the system's states.

Supplementary Note 6: The performance of BR LS in obtaining the elements of the system's density matrix

In the main text, we demonstrate that the bath-driven population transfer can be reproduced by the BR LS approach when a lossy system weakly interacts with a non-Markovian bath. In Fig. S4, we show that BR LS also reproduces the elements of the density matrix: $\langle a^\dagger a\rangle$, $\langle \sigma_+\sigma_-\rangle$ and $\langle \sigma_+ a\rangle$, thus capturing the complete dynamics of the system. Fig. S4 presents the results obtained by the BR, the BR LS, and the numerically exact approaches for the test case given in Fig. 2 in the main text. In (a)–(c), the results for $g_{ec} = 0.1$ eV that yields an on-resonance condition of the transition frequency with the peak in the bath's spectral density $J_b(\omega)$, are presented, and in (d)–(f) the results for

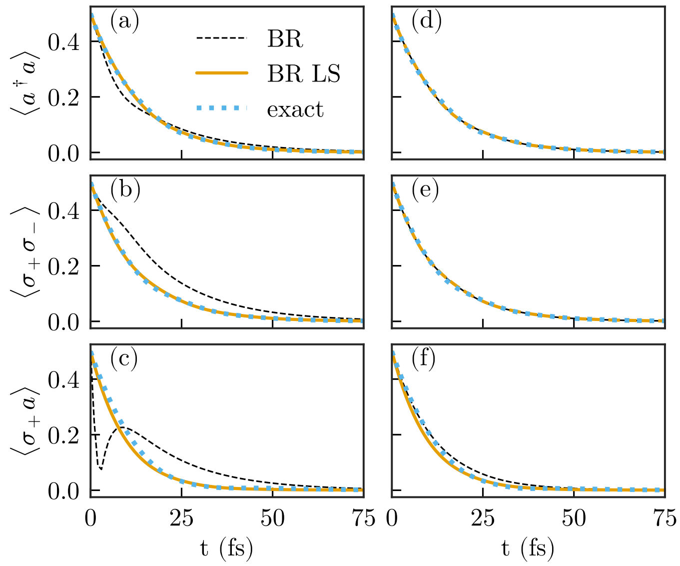

FIG. S4: The density matrix elements obtained by the standard BR approach, the introduced BR LS approach, and numerically exact calculation for the test case studied in Fig. 2 in the main text. (a)–(c) show the results when $g_{ec} = 0.1$ eV, and (d)–(f) when $g_{ec} = 0.11$ eV.

$g_{ec} = 0.11$ eV. All the other parameters are the same as for Fig. 2 in the main text. As can be seen, in both cases, BR LS is in excellent agreement with the numerically exact results and is capable of describing the exact dynamics.

Supplementary Note 7: The performance of BR LS in the case of dephasing

The derivation of BR LS incorporates the effect of the system loss (described by Lindblad superoperators) on the interaction with the non-Markovian bath by using the NH Hamiltonian arising from the coherent terms of the Lindblad superoperators. Thus, although the quantum jump terms in the Lindblad superoperators are taken into consideration in the complete master equation, as shown in Eq. (3) in the main text, BR LS neglects their effect on the interaction with the external bath. Here, however, we demonstrate that this effect is negligible and the broadening of the system's energy levels, described by the NH Hamiltonian, is sufficient to capture the effect of the system loss on the interaction with the other bath, even when the quantum jump terms play a significant role. To do so, we study the test case described in the main text with a dephasing rate rather than decay rates. Namely, we consider the same system's (Hermitian) Hamiltonian as for the test case described in the main text, given by:

$$\hat{H}_s = \omega_c a^\dagger a + \omega_c \sigma_+ \sigma_- + g_{ec}(a^\dagger \sigma_- + \sigma_+ a), \tag{S13}$$

but a different Lindblad superoperator, given by $\gamma_c \hat{L}_{A'=a^\dagger a}$ and describing dephasing rather than decay where $\gamma_c =$ 0.1 eV. The non-Markovian bath's parameters are the same as in the test case in the main text. We present in Fig. S5 the performance of BR LS in reproducing the elements of the density matrix as a function of time when the initial state is $\sigma_+|0\rangle$. For comparison, we also plot the results obtained by the BR and the numerically exact approaches. In (a)–(c), the results obtained for $g_{ec} = 0.1$ eV (on-resonance condition with the bath) are presented, and in (d)–(f) for $g_{ec} = 0.11$ eV (off-resonance condition with the bath). As can be seen, BR LS is in excellent agreement with the numerically exact results. This shows that any interaction with a Markovian bath, even if yielding dephasing rather than decay, can acquire to the system a memoryless character that is captured by the BR LS approach.

Supplementary References

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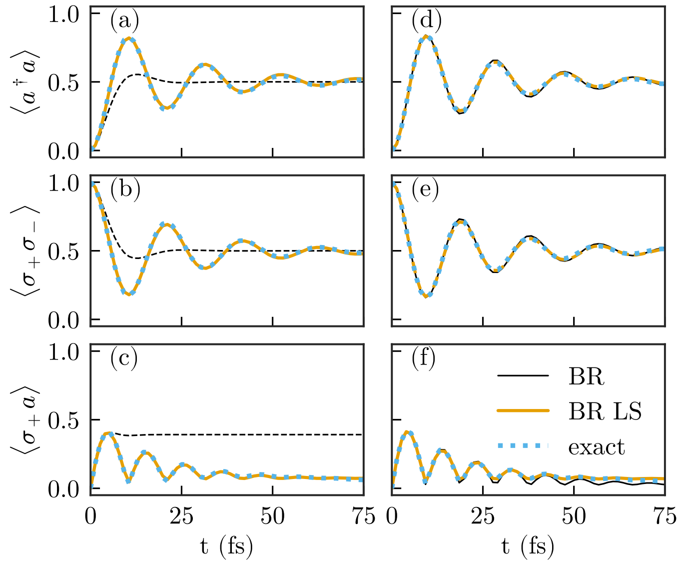

FIG. S5: The density matrix elements obtained by the standard BR approach, the introduced BR LS approach, and numerically exact approach for the case where the Markovian bath results in dephasing. (a)–(c) show the results when $g_{ec} = 0.1$ eV, and (d)–(f) when $g_{ec} = 0.11$ eV.

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