

Supplementary Information for Incorporating External Fields in Analog Ising Machines

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Supplementary Note 1: Simulation of the Ising machine

Simulating a run of the Ising machine begins by drawing the initial spin amplitudes from the same Gaussian distribution as the noise (see further). Their amplitudes are updated each time step according to the Euler scheme:

$$\mathbf{s}_{t+1} = \mathbf{s}_t + \Delta t (-\mathbf{s}_t + \tanh(\alpha \mathbf{s}_t + \beta_t \mathbf{I}_t)) + \gamma \boldsymbol{\xi}_t, \quad (\text{S.1})$$

where $\mathbf{s}_t$ is the vector of all spin amplitudes at time t , $\Delta t = 0.01$ is the Euler time step, α is the linear gain, $\mathbf{I}_t$ is the vector of the local fields of all spin amplitudes at time t , $\gamma = 0.001$ is the noise strength and $\boldsymbol{\xi}_t$ is the vector of the noise terms of all spin amplitudes. Each noise term is drawn from a Gaussian distribution with zero mean and a standard deviation of $\sqrt{\Delta t}$. β_t is the interaction strength at time t and is given by

$$\beta_{t+1} = \beta_t + v_\beta \Delta t, \quad (\text{S.2})$$

where v_β is the annealing speed. The initial β value is β_0 and depends on the method used to implement the

external fields. For the original external fields and the spin sign method, $\beta_0 = 0$ is used as the origin, i.e. all spin amplitudes equal to zero, is a fixed point only at $\beta = 0$. For the auxiliary spin method, β_0 is set to the value at which the origin undergoes its first pitchfork bifurcation, as described in Supplementary Note 4. Because computing this bifurcation point when using the mean absolute spin method is computationally expensive, we also set $\beta_0 = 0$ in that case. More details can be found in Supplementary Note 3.

The updates in Eq. (S.1) and Eq. (S.2) are repeated until either the IM obtains the target energy or $10^4/\Delta t$ updates are done.

For each problem class, the TTS was calculated for all possible combinations of parameter values within the ranges specified in Supplementary Table 1 via a grid search. The number of points per parameter in the last column determines the resolution of the sweep. The TTS mentioned in the main text is the lowest TTS obtained in this way.

As discussed in the main text, Max-3-Cut problems involve an additional parameter, B/A , which controls the relative importance of the edge constraints (weighted by B) compared to the one-hot encoding constraints (weighted by A). The term proportional to A is a sum over all vertices and therefore scales with the number of vertices N_{vert} , while the term

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proportional to B scales with the number of edges, i.e., $\propto N_{\text{vert}}^2$. Consequently, the same relative importance between constraints is achieved at lower values of B/A as N_{vert} increases. Since we use a fixed number of sampling points (7) in our parameter scan, we rescale the range of B/A with $1/N_{\text{vert}}$ to ensure that the scanned values remain relevant and adequately capture the region of interest for each value of N_{vert} .

Supplementary Note 2: Initial search direction without rescaling of the external fields

When using the original external fields method of Eq. 3 of the main paper, the time evolution of the spin amplitudes when using the tanh nonlinearity is given by

$$\frac{ds_i}{dt} = -s_i + \tanh \left(\alpha s_i + \beta \left(\sum_j J_{ij} s_j + h_i \right) \right). \quad (\text{S.3})$$

When $\beta = 0$, the origin (all spin amplitudes equal to zero) is a fixed point of the system. We are now interested in the state that emerges when β is slightly increased. This means that at a fixed point, for all i

$$\beta = \frac{-\alpha s_i + \tanh^{-1}(s_i)}{\sum_j J_{ij} s_j + h_i}. \quad (\text{S.4})$$

The partial derivative with respect to s_k , evaluated at the origin is given by

$$\left. \frac{\partial \beta}{\partial s_k} \right|_{\mathbf{s}=\mathbf{0}, \beta=0} = \frac{1-\alpha}{h_k}. \quad (\text{S.5})$$

Since Eq. (S.4) is invertible for $\alpha < 1$, we can write:

$$\left. \frac{\partial s_k}{\partial \beta} \right|_{\mathbf{s}=\mathbf{0}, \beta=0} = \frac{h_k}{1-\alpha}. \quad (\text{S.6})$$

This means that, when starting from the origin and increasing β , the system will evolve in a direction proportional to the external field h_i , independent of the values of the spin couplings J_{ij} .

Supplementary Note 3: Bifurcation of the mean absolute spin method

In general, the stability of a fixed point is related to the eigenvalues of the Jacobian matrix evaluated at that fixed point. When all eigenvalues are negative, the fixed point is stable. When one eigenvalue becomes positive, the fixed point becomes unstable in the direction of the corresponding eigenvector.

When using the mean absolute spin method in combination with the tanh nonlinearity, the evolution of the spin amplitudes is given by:

$$\frac{ds_i}{dt} = -s_i + \tanh \left(\alpha s_i + \beta \left(\sum_j J_{ij} s_j + h_i \frac{1}{N} \sum_k |s_k| \right) \right). \quad (\text{S.7})$$

The origin bifurcates when its stability changes, i.e. when the largest eigenvalue of the Jacobian matrix goes through zero and becomes positive. The Jacobian matrix evaluated at the origin is given by

$$\begin{aligned} \mathcal{U}_{il} \Big|_{\mathbf{s}=\mathbf{0}} &= - \left. \frac{\partial \frac{ds_i}{dt}}{\partial s_l} \right|_{\mathbf{s}=\mathbf{0}} \\ &= (\alpha - 1)\delta_{il} + \beta J_{il} + \beta \frac{h_l}{N} \lim_{s_l \rightarrow 0} \text{sign}(s_l), \end{aligned} \quad (\text{S.8})$$

where δ_{il} is the Kronecker delta. So, the Jacobian matrix when using the mean absolute spin method is discontinuous at the origin. This means that, in each of the 2^N directions around the origin, the Jacobian has different eigenvalues and eigenvectors. Therefore, to locate the bifurcation point of the origin, one must substitute the signs of the spins of all directions in Eq. (S.8) and determine whether the principal eigenvector has the same signs. The origin will bifurcate in the direction of such an eigenvector with the eigenvalue that first becomes positive. Because of this, determining the β value of the bifurcation point of the origin when using the mean absolute spin method is computationally more demanding than finding the ground state by a brute force search, which defeats the purpose of using an Ising machine.

As the bifurcation point cannot be easily found, the annealing technique is initialized at $\beta_0 = 0$. At

Supplementary Table 1: Parameter sweeps. The ranges of the hyperparameters used to determine their optimal values.

Problem class	Parameter name	Lowest value	Highest value	Spacing type	Number of points
SK	α	-10	1	Linear	12
	v_β	10^{-5}	10^{-1}	Log (base 10)	5
	ζ	0	2	Linear	11
Beasley	α	-10	1	Linear	12
	v_β	10^{-5}	10^{-1}	Log (base 10)	5
	ζ	0	2	Linear	11
Max-3-Cut	α	-10	1	Linear	5
	v_β	10^{-5}	10^{-1}	Log (base 10)	5
	ζ	0	1.2	Linear	7
	B/A	0	$60/N_{\text{vert}}$	Linear	7

this point, however, the origin is still stable in all directions and the spin amplitudes will just remain at zero. The spin amplitudes will only deviate from zero when β has been annealed to a value high enough so that the origin is unstable in at least one direction. This means that the inability to analytically calculate the bifurcation point of the mean absolute spin method leads to wasted simulation time and therefore a slightly inflated TTS.

While it is, in principle, possible to develop a heuristic to estimate an approximate β_0 for the mean absolute spin method—potentially improving its performance—we do not pursue that direction here. Instead, we consider a complementary scenario: setting $\beta_0 = 0$ for the auxiliary spin method, rather than using the value defined in Eq. (S.13) (the latter equation is used to obtain the results in the main text). This intentionally introduces an amount of wasted simulation time, similar to that of the mean absolute spin method, allowing us to assess the impact of β_0 on the TTS of the IM.

Supplementary Fig. 1 illustrates this scenario for (a) SK problems, (b) Beasley problems, and (c) Max-3-Cut problems. It mirrors Figs. 2(b), 3(b), and 7(b) from the main text, but with the data shifted horizontally to the right to reflect the new initialization

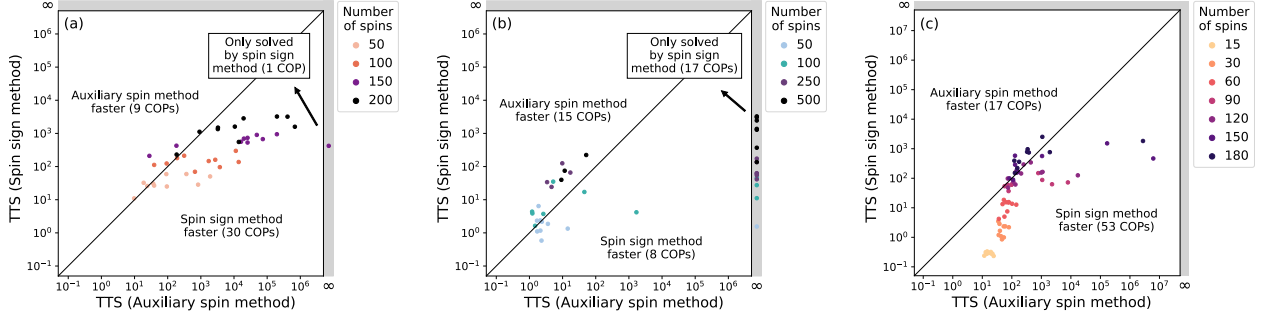
at $\beta_0 = 0$ for the auxiliary spin method. Comparing Supplementary Fig. 1 with the original figures, we find that this shift modestly affects the SK problem data, while its impact on the Beasley and Max-3-Cut problems is negligible. Importantly, even for the SK problems, the shift is small enough that it does not alter the relative performance between methods shown in Fig. 2 of the main text. From these observations we conclude that setting $\beta_0 = 0$ for the mean absolute spin method is not the main cause of the lower performance of this method.

Supplementary Note 4: Bifurcation of the auxiliary spin method

When using the auxiliary spin method in combination with the tanh nonlinearity, the evolution of the spin amplitudes is given by:

$$\frac{ds_i}{dt} = -s_i + \tanh\left(\alpha s_i + \beta \sum_j J'_{ij} s_j\right), \quad (\text{S.9})$$

where $\mathbf{J}'$ is the extended interaction matrix introduced in Eq. (7) of the main text. The Jacobian



Supplementary Figure 1: Comparison of time-to-solution between the auxiliary spin method and the spin sign method. This comparison is made for (a) SK problems, (b) Beasley problems, and (c) Max-3-Cut problems. These plots replicate Figs. 2(b), 3(b), and 7(b) from the main text, but with the auxiliary spin method initialized at $\beta_0 = 0$ instead of the value given in Eq. (S.13). Comparison with the original figures shows that the choice of β_0 has a limited effect on the TTS and does not alter the relative performance between methods.

matrix evaluated at the origin is given by

$$\mathcal{U}_{il} \Big|_{\mathbf{s}=0} = - \frac{\partial \frac{ds_i}{dt}}{\partial s_l} \Big|_{\mathbf{s}=0} = (\alpha - 1)\delta_{il} + \beta J'_{il}. \quad (\text{S.10})$$

The eigenvalues of this matrix λ_i are given by

$$\lambda_i = \alpha - 1 + \beta \mu_i, \quad (\text{S.11})$$

where μ_i are the eigenvalues of the extended interaction matrix $\mathbf{J}'$. When all eigenvalues λ_i are negative, the origin is stable. When one eigenvalue λ_i becomes positive, the origin becomes unstable in the direction of the corresponding eigenvector. This happens when

$$\lambda_+ = 0 = \alpha - 1 + \beta \mu_+ \quad (\text{S.12})$$

$$\beta = \frac{1 - \alpha}{\mu_+}, \quad (\text{S.13})$$

where μ_+ is the largest eigenvalue of the extended interaction matrix $\mathbf{J}'$.

Supplementary Note 5: PySA operation

The PySA algorithm performs simulated annealing with parallel tempering on n_{replicas} replicas at different temperatures. These replicas are initialized in

random states and at a temperature given by

$$T_k = \begin{cases} T_{\min} R^{k-1} & k = 1, \dots, n_{\text{replicas}} - 1 \\ 0 & k = n_{\text{replicas}} \end{cases}$$

where T_k is the temperature of replica k and R is given by

$$R = \exp \left(\frac{1}{n_{\text{replicas}} - 2} \log \left(\frac{T_{\max}}{T_{\min}} \right) \right).$$

So, there is one replica at temperature zero and the other replicas have temperatures distributed between $T_{\min}$ and $T_{\max}$.

The replicas are then updated in sweeps. In our case, a sweep consists of attempting a spin flip for each spin in random order. In total n_{sweeps} are performed for each replica.

This procedure of initializing and performing sweeps is repeated n_{reads} times.

In order to search for the ground state energies of the Sherrington-Kirkpatrick (SK) spin glasses, the algorithm was run five times, using different values for the parameters. The values of these parameters can be found in Supplementary Table 2. This resulted in the same energies that were obtained by the Ising machine simulations for problems of 100 and 150 spins. However, for the problems with 200 spins, the Ising machine found equal or lower energies than PySA.

Supplementary Table 2: PySA parameters. Parameter values used to search for the ground state energy of SK problems.

Run	n_{sweeps}	n_{replicas}	n_{reads}	T_{min}	T_{max}
1	100	5000	1000	0.01	3.0
2	1000	500	1000	0.01	3.0
3	500	500	1000	0.001	6.0
4	500	1000	500	0.001	10.0
5	500	1000	800	0.001	15.0

Supplementary Note 6: Optimal ζ for SK and Beasley problems

As discussed in the Methods section of the main text, setting $\zeta = 1$ guarantees that the ground state of the binary Hamiltonian solves the original COP. This guarantee is lost for $\zeta \neq 1$. However, small deviations from $\zeta = 1$ can still be expected to provide valid solutions in many cases. Moreover, adding ζ as an additional tunable parameter cannot deteriorate the performance. Supplementary Figs. 2 and 3 examine whether such improvements are practically meaningful. Supplementary Fig. 2 compares the TTS obtained using an optimized ζ with the TTS using fixed $\zeta = 1$ for SK problems; Supplementary Fig. 3 shows the same comparison for Beasley instances. In both cases, when methods other than the original external fields are used, most data points lie close to the diagonal, indicating that optimizing ζ leads to minimal improvement in TTS. This effect is most pronounced for the spin sign method.

Supplementary Note 7: Transformation from binary spins to bipolar spins

When solving the Beasley instances of the BiqMac library, the goal is to find the configuration of binary spins $x_i \in \{0, 1\}$ that minimizes $\mathbf{x}^T Q \mathbf{x}$, with Q a symmetric matrix.

This problem can be rewritten in terms of bipolar

spins $\sigma_i \in \{-1, 1\}$ using the transformation

$$x_i = \frac{\sigma_i + 1}{2}. \quad (\text{S.14})$$

The objective function now becomes

$$\sum_{ij} Q_{ij} x_i x_j = \sum_{ij} Q_{ij} \frac{\sigma_i + 1}{2} \frac{\sigma_j + 1}{2} \quad (\text{S.15})$$

$$= \frac{1}{4} \sum_{ij} Q_{ij} (\sigma_i \sigma_j + \sigma_i + \sigma_j + 1). \quad (\text{S.16})$$

Using that Q is symmetric

$$\sum_{ij} Q_{ij} \sigma_i = \sum_{ij} Q_{ji} \sigma_j = \sum_{ij} Q_{ij} \sigma_j, \quad (\text{S.17})$$

We can rewrite the objective function with the following definitions: $J_{ij} := -\frac{1}{2} \tilde{Q}_{ij}$ and $h_i := -\frac{1}{2} \sum_j Q_{ij}$, where $\tilde{Q}$ is the same as Q , but with the diagonal elements set to zero:

$$-\frac{1}{2} \sum_{ij} J_{ij} \sigma_i \sigma_j - \sum_i h_i \sigma_i. \quad (\text{S.18})$$

Here, we neglected the constant term and the diagonal elements of the interaction matrix, as they are of no importance when minimizing the objective function.

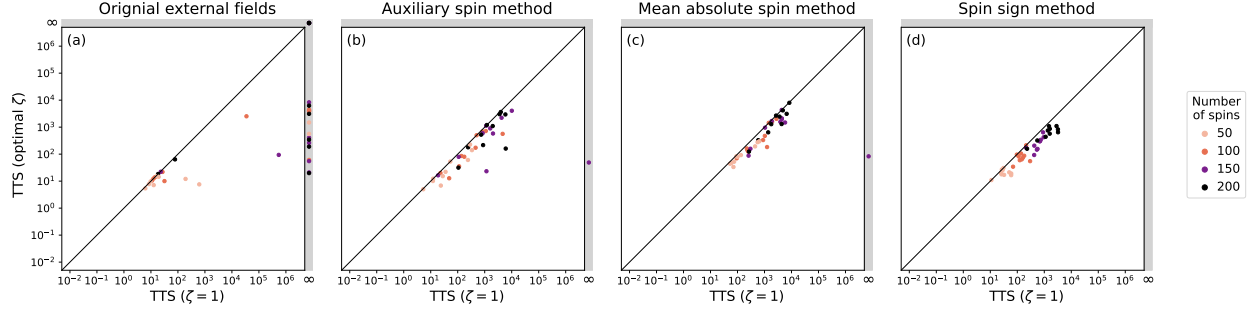
Note that the problem in terms of binary spins does not have any external fields, but the same problem in terms of bipolar spins does.

When tackling Max-3-Cut problems, a similar procedure can be applied. The Hamiltonian of such problems is given by

$$\mathcal{H} = A \sum_{v \in V} \left(1 - \sum_{i=1}^3 x_{v,i} \right)^2 + B \sum_{(uv) \in E} \sum_{i=1}^3 x_{u,i} x_{v,i}, \quad (\text{S.19})$$

This can again be rewritten in terms of bipolar spins by using the transformation of Eq. (S.14).

$$\begin{aligned} \mathcal{H} = A \sum_{v \in V} \left(1 - \sum_{i=1}^3 \frac{\sigma_{v,i} + 1}{2} \right)^2 \\ + B \sum_{(uv) \in E} \sum_{i=1}^3 \frac{\sigma_{u,i} + 1}{2} \frac{\sigma_{v,i} + 1}{2}. \end{aligned}$$



Supplementary Figure 2: Comparison of time-to-solution between $\zeta = 1$ and the optimal ζ that minimizes the TTS, for each of the methods, for SK problems. Panels (a)-(d) correspond to the original external fields, auxiliary spin method, mean absolute spin method, and spin sign method, respectively. In panel (a), 6 data points coincide in the grey area in the top right. The corresponding COPs could not be solved with $\zeta = 1$, nor with optimal ζ (i.e. $TTS = \infty$, $SR = 0\%$), given the allocated compute time of $t = 10^4$.

We can rewrite the first term as follows

$$\begin{aligned}
& A \sum_{v \in V} \left(1 - \sum_{i=1}^3 \frac{\sigma_{v,i} + 1}{2} \right)^2 \\
&= A \sum_{v \in V} \left(1 + \left(\sum_{i=1}^3 \frac{\sigma_{v,i} + 1}{2} \right)^2 - 2 \sum_{i=1}^3 \frac{\sigma_{v,i} + 1}{2} \right) \\
&= A \sum_{v \in V} \left(\sum_{i,j=1}^3 \frac{\sigma_{v,i} \sigma_{v,j}}{4} + \sum_{i=1}^3 \frac{\sigma_{v,i}}{2} + \frac{1}{4} \right).
\end{aligned}$$

The second term can be rewritten as

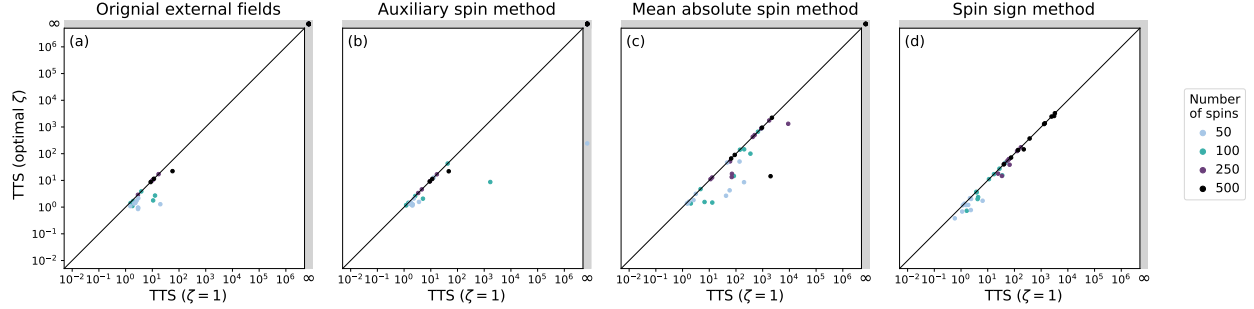
$$\begin{aligned}
& B \sum_{(uv) \in E} \sum_{i=1}^3 \frac{\sigma_{u,i} + 1}{2} \frac{\sigma_{v,i} + 1}{2} \\
&= B \sum_{(uv) \in E} \sum_{i=1}^3 \left(\frac{\sigma_{u,i} \sigma_{v,i}}{4} + \frac{1}{4} \right) \\
&\quad + B \sum_{(uv) \in E} \sum_{i=1}^3 \left(\frac{\sigma_{u,i}}{4} + \frac{\sigma_{v,i}}{4} \right) \\
&= B \sum_{(uv) \in E} \sum_{i=1}^3 \left(\frac{\sigma_{u,i} \sigma_{v,i}}{4} + \frac{1}{4} \right) \\
&\quad + B \sum_{v \in V} \sum_{i=1}^3 \frac{\sigma_{v,i}}{4} \deg(v)
\end{aligned}$$

where in the last line, we changed the sum of the second term from going over all edges of the graph, to going over all vertices. This, while once again ignoring all constant terms gives

$$\begin{aligned}
\mathcal{H} &= \frac{A}{4} \sum_{v \in V} \sum_{i \neq j} \sigma_{v,i} \sigma_{v,j} + \frac{B}{4} \sum_{(uv) \in E} \sum_{i=1}^3 \sigma_{u,i} \sigma_{v,i} \\
&\quad + \sum_{v \in V} \sum_{i=1}^3 \left(\frac{A}{2} + \frac{B}{4} \deg(v) \right) \sigma_{v,i},
\end{aligned} \tag{S.20}$$

Supplementary Note 8: Optimal ζ for Max-3-Cut problems

As explained in the main text, the optimal ζ value for Max-3-Cut problems converges to 0.6 as the problem size increases. While there is some variation on this problem-specific optimal ζ value, using $\zeta = 0.6$ yields comparable performance across all problems and methods of implementing the external fields. Supplementary Fig. 4 compares the time-to-solution (TTS) achieved using the optimized value of ζ (vertical axis) versus a fixed value of ζ (horizontal axis) for all Max-3-Cut problem instances. Each row corresponds to a different fixed value: $\zeta = 0.4$, $\zeta = 0.6$,



Supplementary Figure 3: Comparison of time-to-solution between $\zeta = 1$ and the optimal ζ that minimizes the TTS, for each of the methods, for Beasley problems. If COPs could not be solved with $\zeta = 1$, nor with optimal ζ (i.e. $TTS = \infty$, $SR = 0\%$), given the allocated compute time of $t = 10^4$, then they appear in the grey area at the top right. These data points coincide. The number of such COPs in Fig. (a)–(d) is 17, 16, 5, and 0, respectively.

and $\zeta = 0.8$ and each column represents one of the four methods considered to implement external fields. In these plots, each point represents a single problem instance. Points lying on the diagonal indicate that either the fixed value of ζ is the instance-specific optimal value or that it has a similar performance. Notably, for $\zeta = 0.6$, the majority of points cluster tightly around the diagonal across all methods, suggesting that this choice often yields near-optimal performance. In contrast, for $\zeta = 0.4$ and $\zeta = 0.8$, many points fall noticeably below the diagonal, indicating that fixing ζ to these values more frequently results in suboptimal performance.

Supplementary Note 9: Extending the connection between ζ and the embedding validity to larger problems

Due to limited computational resources, it is not feasible to generate mapping-correctness plots like Fig. 6 from the main text for systems larger than 30 spins. However, as the IM is only expected to perform well when the mapping is correct, regions of high success rate can serve as a proxy for identifying regions of valid mapping. As illustrated in Supplementary

Fig. 5, the success rate under slow annealing conditions strongly correlates with the correct-mapping region identified in Fig. 6 of the main text. Notably, using $\zeta = 0.4$ or $\zeta = 0.6$ allows for a broader range of B/A values that yield high success rates. To quantify this trend, Supplementary Fig. 6 shows the fraction of tested B/A values (defined in Supplementary Table 1) that result in success rates larger than 10% as a function of system size. The figure reveals that this fraction shrinks with increasing problem size for $\zeta = 1$, while such a strong decrease is not observed for $\zeta = 0.6$. Even though these figures are made using the original external fields, the conclusions also hold for the other methods, as the mapping correctness is independent of the method of implementing the external field. Moreover, the conclusions also hold when varying the arbitrary threshold of 10% success rate.

Supplementary Note 10: Optimal working point for the spin sign method.

Supplementary Figs. 7–9 present the success rate, TTS, and average runtime T of successful runs for all problem instances. Supplementary Fig. 7 corresponds to the SK instances, where the number following “N” in the label indicates the number of spins.

Supplementary Fig. 8 and Supplementary Fig. 9 report results for the Beasley and Max-3-Cut problems, respectively, using the BiqMac naming convention.

Across these figures, the spin-sign method often exhibits substantially lower success rates than the other three methods. However, its significantly shorter average runtime frequently results in superior TTS performance. This is because, when optimizing for TTS, the hyperparameter optimization tends to favor scenarios in which the spin-sign method rapidly converges, albeit with lower success probabilities, leading to a lower overall TTS. This type of operation is sometimes referred to as *fail fast mode* [1]. Note that it is more often observed for smaller problem sizes. For larger problems, T tends to be more similar between all methods.

It is also important to note that the spin-sign method does not universally underperform in terms of success rate. While its success rates can be lower when performing a hyperparameter optimization that minimizes TTS, this is a consequence of favoring fast-converging configurations rather than a fundamental limitation of the method. In fact, several instances are solvable only with this method, while the others fail. Furthermore, as shown in Supplementary Fig. 10, when the hyperparameters are optimized to maximize success rate instead of TTS, the spin-sign method operates in a regime with consistently higher success rates than before, often comparable to or higher than those of the other methods.

Supplementary Note 11: Maximal cut-values for Max-3-Cut problems from the Gset library.

In Fig. 7 of the main text, we showed that the spin-sign method successfully solves the Max-3-Cut problem on the first five graphs of the Gset benchmark set. The other methods generally fail to do so, with one exception: the auxiliary spin method succeeds on the G4 instance but fails on the others. For each method, Supplementary Table 3 lists the maximal cut values obtained after hyperparameter optimization (cf. Supplementary Note 1).

Supplementary Note 12: Exact vs. approximate balance between spin couplings and external fields

In the Methods section of the main text, we introduced different methods for incorporating external fields in analog IMs and illustrated their impact on spin dynamics using a 3-spin COP. The original external fields showed to be prone to magnitude imbalances between the spin couplings J_{ij} and external fields h_i , most clearly reflected in the initial search direction, which depends exclusively on h_i and not on J_{ij} (cf. Supplementary Note 2). To reduce this imbalance, the mean absolute spin and auxiliary spin methods apply a continuously varying correction factor to the external fields. This factor mitigates, but does not eliminate, the discrepancy.

The spin sign method, in contrast, preserves the exact magnitudes of all interactions as defined in the binary Hamiltonian (Eq. 1 of the main text). As a result, it reproduces the intended energy landscape more faithfully and governs spin evolution accordingly. To formalize this comparison, we define the error metric ϵ_i , which quantifies how much the effective ratio between spin couplings and external fields, as experienced by spin i , deviates from the target ratio set by the Ising Hamiltonian:

$$\epsilon_i = \frac{\left| \frac{\sum_j J_{ij} x_j}{h_i \Gamma} \right|}{\left| \frac{\sum_j J_{ij} \text{sgn}(x_j)}{h_i} \right|} = \left| \frac{\sum_j J_{ij} x_j}{\Gamma \sum_j J_{ij} \text{sgn}(x_j)} \right|. \quad (\text{S.21})$$

Here, Γ denotes the time-dependent correction factor introduced by the methods:

$$\Gamma = \begin{cases} 1, & \text{for the original external fields,} \\ \frac{1}{N} \sum_k |s_k|, & \text{for the mean absolute spin method,} \\ x_{\text{aux}}, & \text{for the auxiliary spin method.} \end{cases} \quad (\text{S.22})$$

By construction, $\epsilon_i = 1$ for all i under the spin sign method.

Supplementary Fig. 11 shows the temporal evolution for the Beasley problem **bpq50-2** under the different methods. Qualitatively similar results were observed for other COPs, confirming that the following conclusions are not specific to this instance.

Supplementary Table 3: Maximum cut for GSET. Maximal cut-values for the first five graphs of the Gset benchmarks using the different methods under consideration. Values in bold highlight where the best known solution is obtained.

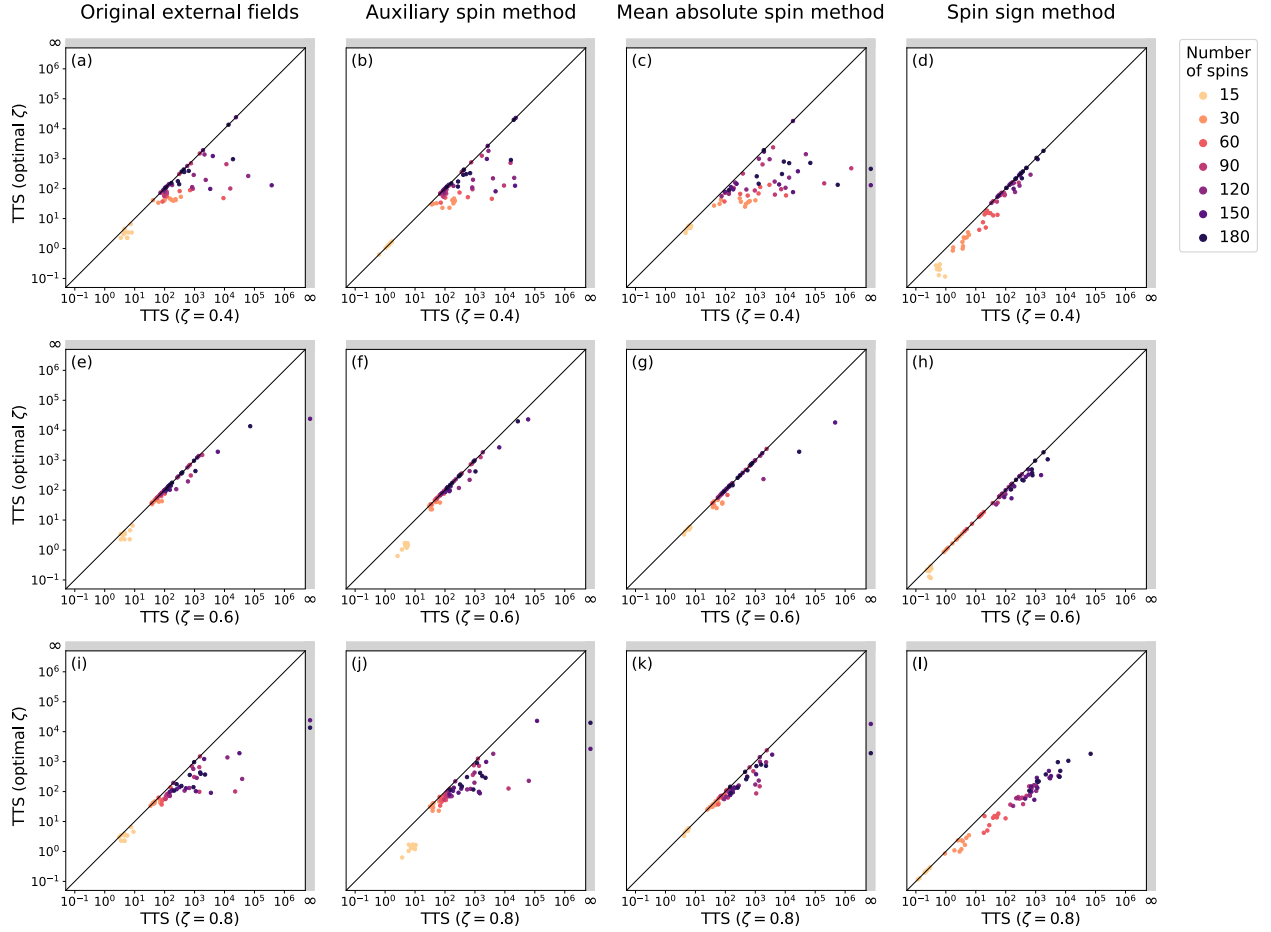
Graph	Best known solution [2]	Spin sign method	Original external fields	Auxiliary spin method	Mean absolute spin method
G1	15165	15165	15160	15159	15156
G2	15172	15172	15164	15165	15162
G3	15173	15173	15166	15168	15168
G4	15184	15184	15178	15184	15174
G5	15193	15193	15189	15191	15191

Panels (a–d) display the spin amplitudes, which generally converge to ± 1 , although a subset of spins remain frustrated and fail to do so. Panels (e–h) report the problem-solving progress, showing that all methods eventually reach the solution but require different operating times. Note that frustrated spins with zero amplitude may persist even when the IM energy coincides with the ground state, indicating that the ground state energy is achieved regardless of whether those spins map to $+1$ or -1 . Panels (i–l) present the error metric ϵ_i . For the original external fields of panel (i), ϵ_i starts near zero, reflecting that the initial search direction is dominated by h_i and independent of J_{ij} (cf. Supplementary Note 2). As the IM evolves, ϵ_i exhibits irregular fluctuations and deviates from unity. This indicates that the intended binary energy landscape is not faithfully represented during the active search. As time progresses even further, the spin amplitudes generally converge to ± 1 . As is clear from Eq. (S.21), this makes most ϵ_i approach 1 (although some remain offset due to spin frustrations). While this indicates a more faithful representation of the intended energy landscape, it occurs only after the spins have effectively “frozen” due to convergence, and therefore does not reflect accurate behavior during the search phase. Considering panels (j) and (k), we observe that both the mean absolute spin method and the auxiliary spin method do not suffer from a field-dominated initial search direction, such that $\epsilon_i > 0$ at $t = 0$. However, during the remainder of the evolution ($t > 0$), we still ob-

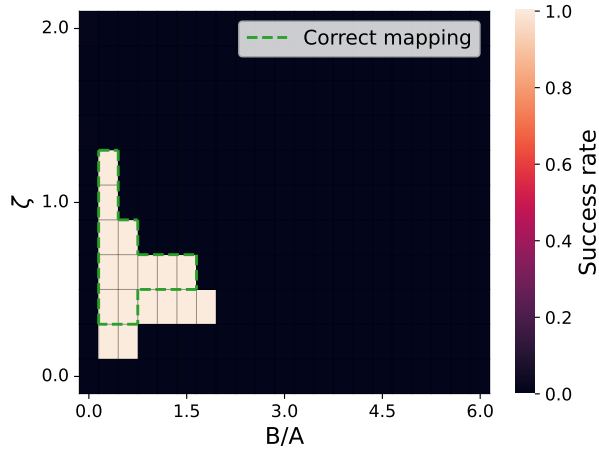
serve erratic fluctuations that differ from unity. This indicates that, even though these two methods aim to assess the imbalances between external fields and spin couplings using a global continuously-varying Γ -factor, they cannot enforce the exact intended ratio for each spin. The spin sign method, in contrast, maintains exact consistency with the relative magnitudes of all terms in the binary Hamiltonian and thus provides a more faithful representation of the energy landscape governing spin evolution.

Supplementary References

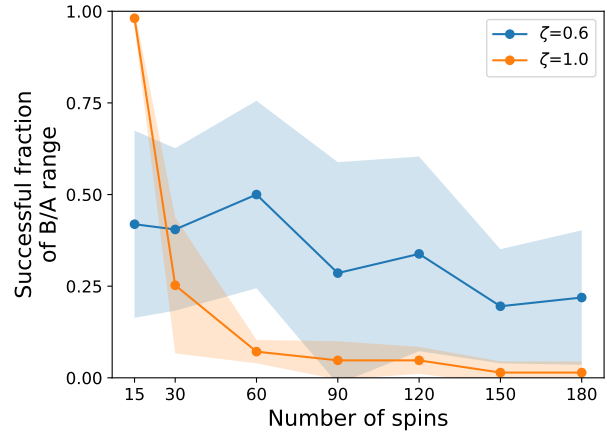
- [1] Moslem Noori, Elisabetta Valiante, Thomas Van Vaerenbergh, Masoud Mohseni, and Ignacio Rozada. A statistical analysis for per-instance evaluation of stochastic optimizers: How many repeats are enough? *arXiv preprint arXiv:2503.16589*, 2025.
- [2] Fuda Ma and Jin-Kao Hao. A multiple search operator heuristic for the max-k-cut problem. *Annals of Operations Research*, 248(1):365–403, 2017.



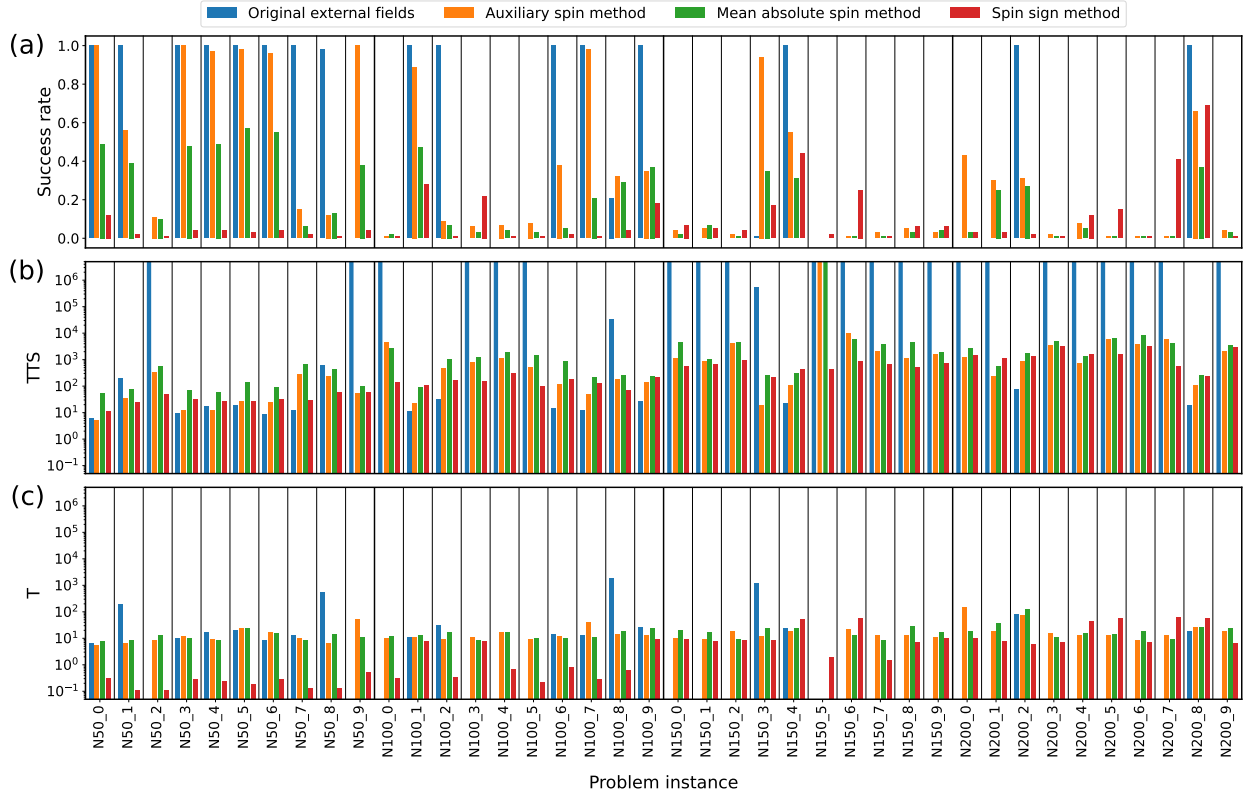
Supplementary Figure 4: Comparison of the TTS achieved with optimized ζ values versus fixed $\zeta \in \{0.4, 0.6, 0.8\}$ for Max-3-Cut instances. Each column corresponds to a method to implement external fields: the original external fields (panels a,e,i), the auxiliary spin method (panels b,f,j), the mean absolute spin method (panels c,g,k), and the spin sign method (panels d,h,l). The vertical axis of each panel shows the TTS obtained using the optimal ζ per instance, while the horizontal axis shows the TTS with a fixed value of ζ . Panels (a–d), (e–h), and (i–l) correspond to fixed ζ values of 0.4, 0.6, and 0.8, respectively. Each point represents a single problem instance. Points lying on the diagonal indicate equal performance between fixed and optimized ζ .



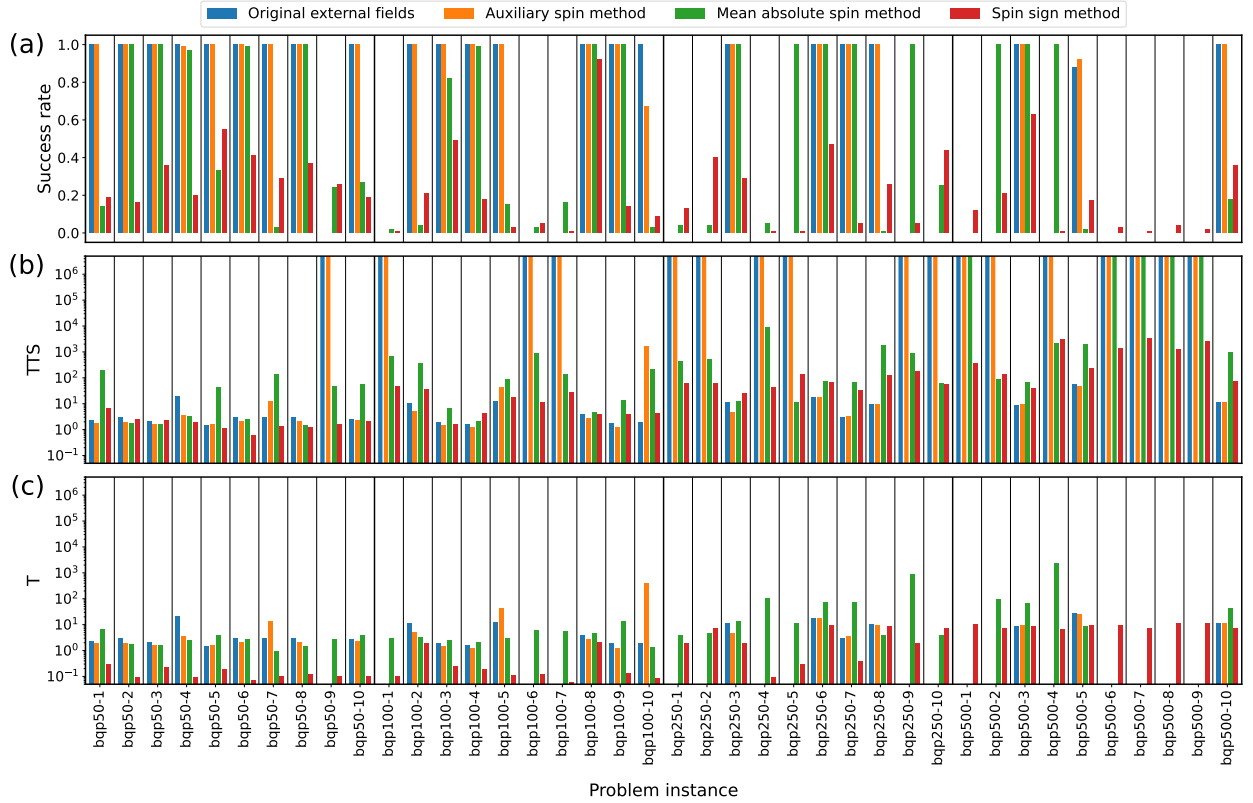
Supplementary Figure 5: Correlation between success rate and correct mapping. Success rate of the g05_10.0 graph when rescaling the external fields with a constant ζ , as a function of ζ and B/A when slow annealing is performed ($d\beta/dt = 0.001$). No dynamic rescaling factor is applied. Other parameters are set as follows: $\alpha = -1$, $\gamma = 0.001$, $n_{\text{init}} = 100$.



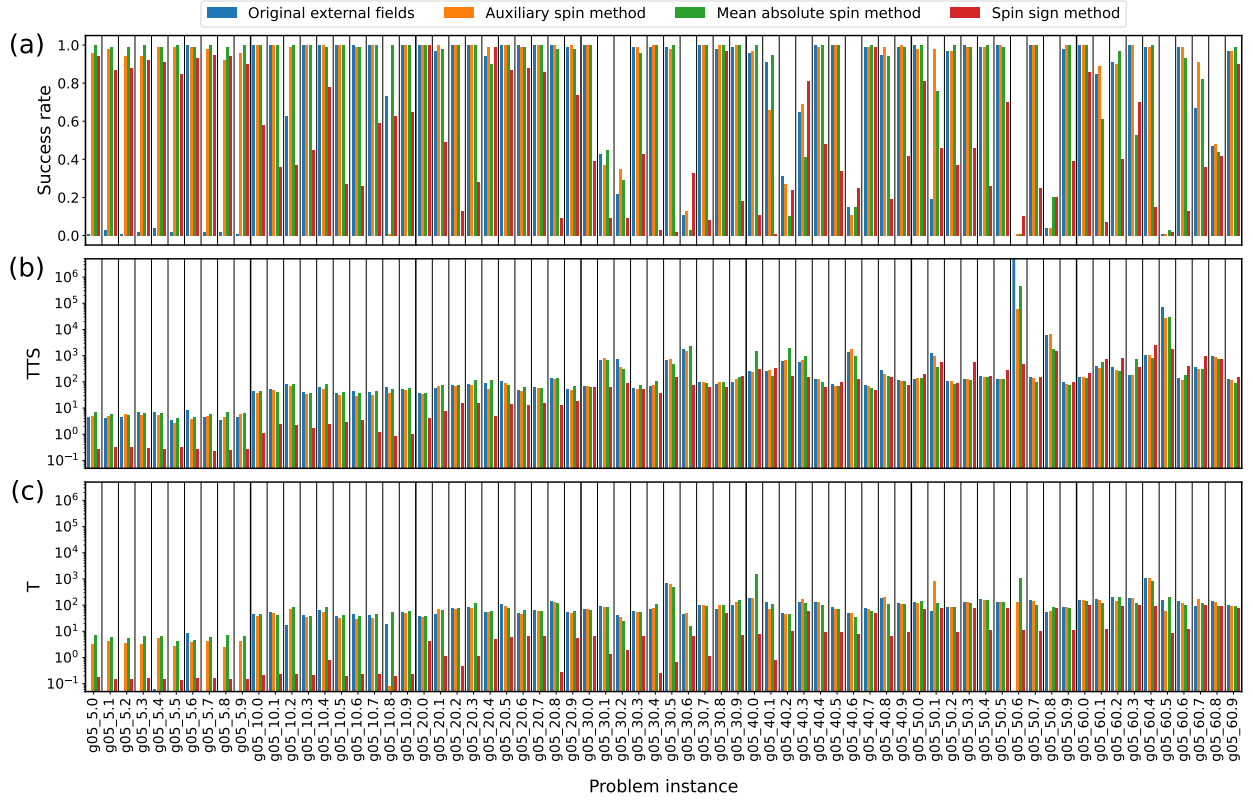
Supplementary Figure 6: Fraction of valid B/A values. Fraction of the tested $\frac{B}{A}$ values (defined in Supplemantray Table 1) for which the success rate exceeds an arbitrary threshold of 10% as a function of the number of spins. Each dot represents the mean value across 10 Max-3-Cut problems. Shaded areas represent the standard deviation. The underlying simulations were performed for the original external fields method, $\alpha = -1$, and $\frac{d\beta}{dt} = 0.001$.



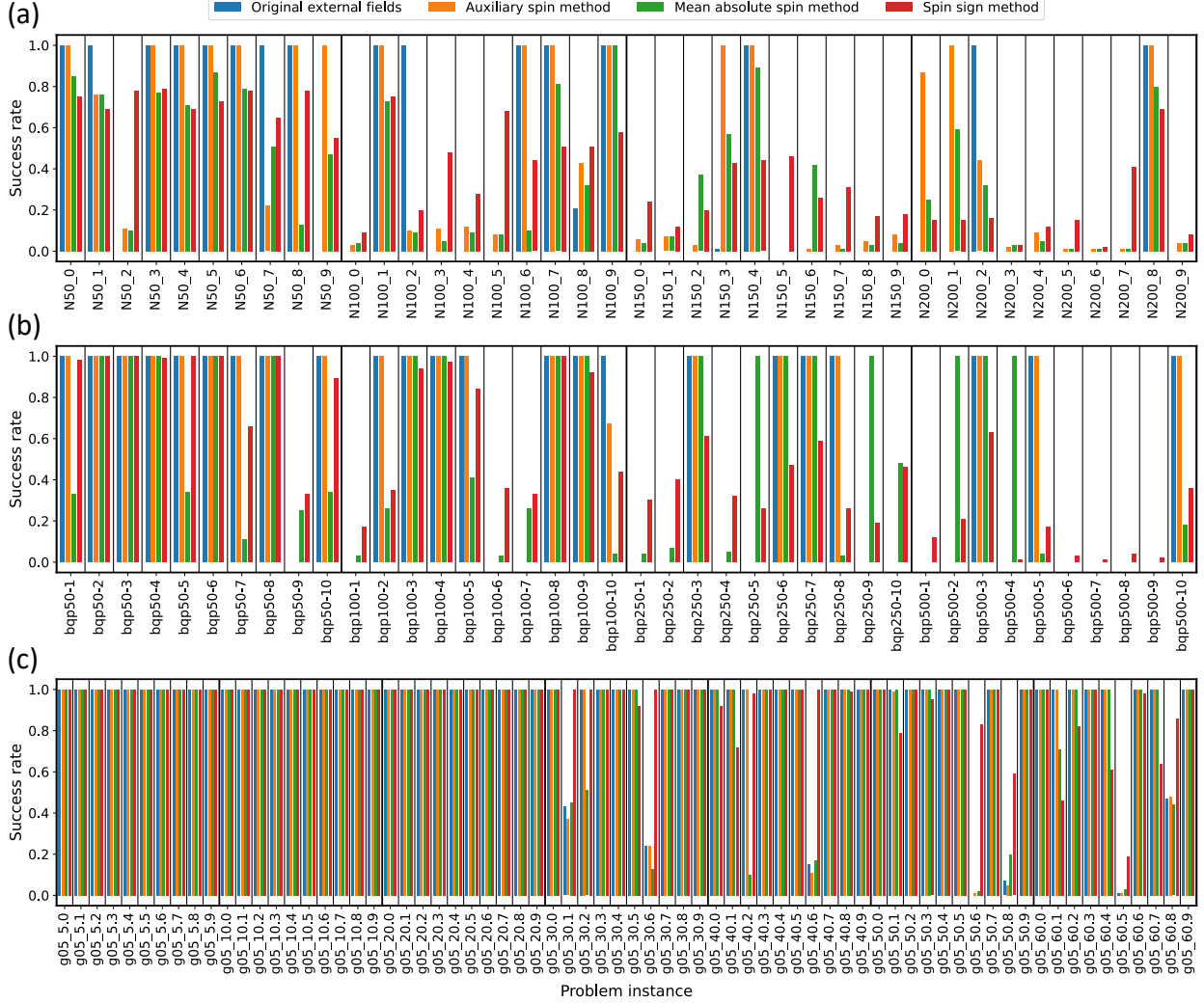
Supplementary Figure 7: Performance for SK problems. For every SK problem instance and for every implementation method for the external fields, the values of α and v_β are determined that minimize TTS. For those values of the hyperparameters, the success rate (panel a), time-to-solution (TTS, panel b), and average runtime T of successful runs (panel c) is shown. TTS bars that reach the upper edge denote TTS= ∞ .



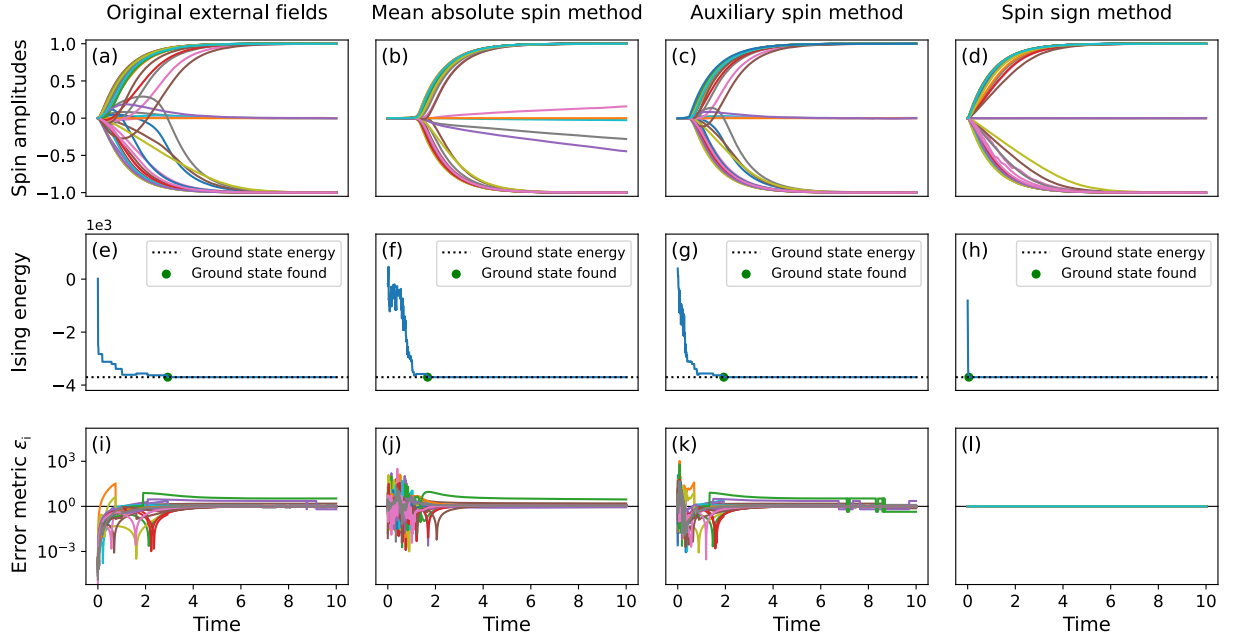
Supplementary Figure 8: Performance for Beasley problems. For every Beasley problem instance and for every implementation method for the external fields, the values of α and v_β are determined that minimize TTS. For those values of the hyperparameters, the success rate (panel a), time-to-solution (TTS, panel b), and average runtime T of successful runs (panel c) is shown. TTS bars that reach the upper edge denote $TTS = \infty$.



Supplementary Figure 9: Performance for Max-3-Cut problems. For every Max-3-Cut problem instance and for every implementation method for the external fields, the values of α and v_β are determined that minimize TTS. For those values of the hyperparameters, the success rate (panel a), time-to-solution (TTS, panel b), and average runtime T of successful runs (panel c) is shown. TTS bars that reach the upper edge denote TTS= ∞ .



Supplementary Figure 10: Performance for all problems when prioritizing success rate. Results with hyperparameters chosen to maximize success rate. This contrasts with the main text, where hyperparameters are chosen to minimize TTS. Panels (a) and (b) show $\max_{\alpha, v_\beta}$ SR for SK and Beasley problems, respectively, with $\zeta = 1$. Panel (c) shows $\max_{\alpha, v_\beta, \frac{B}{A}}$ SR for Max-3-Cut problems, with $\zeta = 0.6$.



Supplementary Figure 11: Temporal evolution of the spin amplitudes, Ising energy and error metric. Temporal evolution for the Beasley problem bq50-2 for the different methods under consideration. (a–d) Spin amplitudes generally converge to ± 1 , though some spins remain frustrated. (e–h) All methods eventually reach the solution, with varying operating times. (i–l) The error metric ϵ_i fluctuates for the original external fields, mean absolute spin, and auxiliary spin methods, while remaining exactly 1 for the spin sign method. This highlights that only the spin sign method preserves the intended balance between spin couplings and external fields, faithfully representing the binary Hamiltonian.