nature portfolio

Peer Review File

Geometric Features of Higher-Order Networks via the Spectral Triplet

Corresponding Author: Professor Sara Najem

This file contains all reviewer reports in order by version, followed by all author rebuttals in order by version.

Version 0:

Reviewer comments:

Reviewer #1

(Remarks to the Author)
The present manuscript applied a well known theory in the field of mathematical physics associated to non-commutative algebras to the algebra associated with their matrix representation of simplicial complexes. Then they applaid this mathematical framework to real data set of musical composition and revealing geometric structures. Although the present study has some potential interest, I can not recommend publication in the present form and an extensive revision and rewriting of the manuscript is mandatory.

Major concerns

The present manuscript applies a well-known theory in mathematical physics - namely, the framework of non-commutative algebras - to the algebra associated with the matrix representation of simplicial complexes. The authors then apply this mathematical framework to a real dataset of musical compositions, aiming to reveal underlying geometric structures. Although the study may have some potential interest, I cannot recommend publication in its present form, and an extensive revision and rewriting of the manuscript is required.

Major concerns

1. The paper is written in poor English. For example, the abstract begins with:
"Our work is concerned with..."
I believe it is more formal to write:
"The present work is concerned with..."

Therefore, a thorough and careful reading and rewriting of the manuscript is necessary. There are also many typographical errors. For instance, the following sentence is unclear:

"Also, inspired by Weyl's law governing the spectrum of the Laplace–Beltrami we operator here we us…"

2. The manuscript is too mathematical and formal, making it difficult to follow. It may not be suitable for the general audience of Communications Physics and would likely be more appropriate for a specialized mathematical-physics journal. Furthermore, some mathematical symbols are not defined, and the authors refer to existing literature for fundamental definitions. A paper of this type should be fully self-contained, presenting all relevant mathematical definitions explicitly so that readers do not need to consult external sources to understand the notation.

3. The main contribution of the present work remains unclear. The authors state in the abstract:

"The prime contribution of this work is the introduction of geometric measures for these simplicial complexes."

However, most of the mathematical derivations presented appear to be already available in the literature, including the geometric measures discussed. Moreover, the application of the theoretical framework does not appear to be new, as the authors have previously published results on closely related topics. The authors must clearly state—explicitly in the abstract, introduction, and conclusions what are the original contributions of the present manuscript.

4. The quality and explanations of the figures are poor. Many figures contain an excessive number of tick labels, which makes it difficult to clearly visualize the main findings. The text in the figures is also too small to be easily readable.

Minor points

1. Please revise the English grammar and the typographical errors throughout the manuscript.

2. Many figures are not correctly formatted. For example, some appear in a double-column layout with duplicated captions, which is not appropriate for a single-column manuscript format. I recommend that the authors present such figures as two-panel figures with a single caption that merges the content of the current double captions.

Reviewer #2

(Remarks to the Author)
The idea of applying elements of non-commutative geometry to the student of (higher-order) networks is based on an observation of Requardt ref.47 that the graph metric can be recovered from a spectral triple - first equation in 3.2.1 This is worth studying in the context of higher order networks such as simplicial complexes and hence I welcome this approach even if I'm still unsure of its significance and applicability.

Unfortunately, the manuscript is not very well written at the moment to the high standards that I would expect for this journal. Hence, I recommend that the authors resubmitted a much improved version before I can give it full consideration.

MAJOR CONCERNS
- FORMALITY. Formal, unambiguous mathematical definitions and notation of all key concepts, consistent notation, explaining all symbols used etc see examples below
- READABILITY. Improve readability by explaining the significance of the key concepts used and results from the literature. At the moment is not accessible to a generalist audience. It needs to be much more self-contained if you are addressing the audience of this journal.
- ANALYSIS & INTERPRETATION. In the musical case study, the analysis is mostly descriptive and quite superficial - what does it mean? For example, the captions of all but figure 8 are purely descriptive.
- IMPORTANCE. What is the importance/significance of these results? can they be reproduced with other (simpler) methods? What is the key advantage of this approach and why?
- APPLICABILITY. Can this method be applied to other datasets? If yes, show other examples, if not explain the limitations.

EXAMPLES (non-exhaustive)
p2 higher-order network (hypergraph) 'set of vertices V and hyperedges H' that is not enough - do you allow repetition, is the order important, do you admit weights or only the binary case, what is an orientation, how is it (well) defined, are the elements called hyperedges or simplices, what are delta and d in equation 1, what is the range of n in equation 1

p4 missing D, unexplained notation e.g. t_{ijkl}, H_i, s (what is the codomain), the norm and bracket [D,f] for graphs and higher-order graphs, equation after eq.3, etc

p9 how is the simplicial complex constructed? 'C played followed by [D,E,F], an edge is added between C and the 2-simplex [D,E,F]' how? which node (note) do you connnect C to in [D,E,F]?

p13 the conclusion section mentions things not really investigated: 'performing a random walk'; 'geometric embedding'; 'adapted to these contexts

p18 comparison to pairwise distances - normalised as different ranges otherwise misleading e.g. fig24 vs fig27

Reviewer #3

(Remarks to the Author)
The article “Geometric Features of Higher-Order Networks via the Spectral Triplet”, proposes using Connes’ spectral triple to define dimension, curvature, and distance on simplicial complexes and illustrates this approach on higher-order networks extracted from Bach’s solo violin sonatas and partitas.

Summary and main contributions

* The authors consider higher-order networks modeled as simplicial complexes up to dimension 3 (nodes, edges, triangles, tetrahedra).

* They define the super-Laplacian L_s as the block-diagonal matrix of Hodge Laplacians $L^{[k]}$ and construct a discrete Dirac operator D whose square is L_s.
* Using the eigenvalues of the Hodge Laplacians, they adopt the notion of a vector of spectral dimensions $d_s^{[k]}$, following Torres–Bianconi.
* They propose to define a “spectral curvature” via the spectral action and heat kernel expansion of e^{-tL_s/Λ^2}, with node-level curvature read off from the coefficient $u_1(i, i)$ in the small-t expansion of the diagonal of the kernel.
* They also adapt Connes’ distance to simplicial complexes, imposing constraints on s of a 0-form f on nodes; they compute these distances numerically using CVXR and compare to graph-geodesic distances.
* Finally, they compare their curvature to the Forman–Ricci curvature (Bochner–Weitzenböck decomposition) and interpret patterns in the musical structures.

Overall, the idea of combining non-commutative geometry with higher-order networks is interesting and timely, and the musical case study is appealing. However, there are significant conceptual and mathematical issues that must be addressed, and the exposition needs considerable polishing.

My overall recommendation would be a MAJOR REVISION.

Please find below a list of more detailed comments of mine.

Mathematical and conceptual issues

1. Notation and normalization in the density formula.
The spectral density for $0<n<k$ is written as
$$\rho(\lambda) = \frac{N^{[n-1]}}{N^{[n]}}C^{[n-1]}\lambda^{d_s^{[n-1]}/2} + \frac{N^{[n]}}{N^{[n]}}C^{[n]}\lambda^{d_s^{[n]}/2}$$
The second factor $N^{[n]}/N^{[n]}$ is identically 1 and seems to be a typo; presumably the denominator should be the number of simplices of a different order.
Furthermore, later, the text says “where N is the number of non-zero eigenvalues … and N is the number of simplices of the same order”, using the same symbol for two different quantities. This must be clarified, and the expression checked carefully.

2. Continuum vs discrete scaling.
The derivation of spectral dimensions from the cumulative eigenvalue density should specify the fitting range in λ and the procedure (e.g., log-log fit) used on finite complexes. Right now, the text directly imports the continuum Weyl law and asserts proportionalities without clearly explaining how they are realized in the finite, discrete case.

3. Connes distance and commutator constraints (Sec. 3.2.1 and Appendix)

Misprint in the edge commutator formula.Equation (4) in the main text reads$[B_0,f]\alpha(i,j) = (f_i - f_j)^2(\alpha(i)+\alpha(j))$, while the derivation in the Appendix gives$[B_0,f]\alpha(i,j) = \frac{f_i - f_j}{2}\,(\alpha(i)+\alpha(j))$, and this latter expression is needed to obtain the constraint $\|[D,f]\| = \|(f_i-f_j)/2\| \le 1$.
So either Eq. (4) or the Appendix must be corrected; as written, they are inconsistent, and Eq. (5) in the main text appears to rely on the Appendix form. This is a genuine mathematical error, not just a typo.

4. Higher-order constraints derived from triangles/tetrahedra.

The formulas (6) and (7) for triangles and tetrahedra follow from the chosen averaging rules, but then the argument that summing over the simplex yields zero and hence $\|[B_1,f]\|=0$ and $\|[B_2,f]\|=0$ is too quick. Expand and clarify, please.

The step “Therefore, we enforce the constraint on all pairwise interactions in the simplex $\|(f_i-f_j)/2\|\le 1,\dots$” is not mathematically justified; it is a design choice. This must be clearly stated as an assumption rather than as a consequence of the commutator structure.

5. Norm used in $\|[D,f]\|$.
The operator norm used in the constraint $\|[D,f]\|\le 1$ should be specified precisely (spectral norm? Hilbert-Schmidt?). This matters when passing from matrix inequalities to scalar inequalities on edge s.

6. Generalization of the Connes distance formula.The definition$\operatorname{dist}_{C,N}(i,j) = \sup\{ |f_j-f_i| : \|[D,f]\|\le 1\}$is imported from graphs, but now the Dirac operator acts on a direct sum over 0-, 1-, 2- and 3-simplices. The authors restrict attention to constraints on edges only; this should be argued more carefully (e.g., by showing that higher-order blocks do not change the supremum, or by explaining why it is reasonable to ignore them).

6. Spectral action, heat kernel and curvature (Sec. 3.2.2)
Confusing and probably incorrect use of d_s, D, and Λ.After introducing the spectral action $\mathrm{Tr}\,F(D/\Lambda)$, the manuscript states“where $\{\lambda_i\}$ are the eigenvalues of D, and $d_s = D^{-1} = |1/\Lambda_1| = 1/\Lambda$”Here D^{-1} is an operator, while d_s was previously the spectral dimension vector. Identifying a dimension with the inverse of the smallest eigenvalue is dimensionally and conceptually incorrect. The sentence should be

removed or rewritten entirely; at the moment, it mixes three unrelated notions: spectral dimension, inverse Dirac operator, and cutoff \Lambda.

7. Mismatch of exponents and dimensions in Eq. (12). The diagonal heat kernel expansions are written as (for vertices)K_v(i, i,t) \approx u_{0,v}(i, i)(4\pi t)^{-d_s^{[0]}/2} + u_{1,v}(i, i)(4\pi t)^{-d_s^{[0]}/2}t + O(t^{2-d_s^{[0]}/2}). But in the continuum, the (4\pi t)^{-n/2} prefactor involves the topological (manifold) dimension n, not the spectral dimension. If the authors want to replace n by the empirically fitted d_s^{[0]}, they should justify this substitution; otherwise, the scaling in (12) is not derived but assumed. The same applies to the edge and face formulas.

8. Inconsistent treatment of the (4\pi t)^{-n/2} factor. To reconcile the exact series (14) and the heat-kernel expansion (15), the authors write"we take (4\pi t)^{-n/2} \to 1, which now defines t^+ \to 0^+" Mathematically, as t\to 0^+, (4\pi t)^{-n/2} diverges; it cannot be approximated by 1. One could factor it out and absorb it into the coefficients, or work with a fixed small but finite t, but the present statement is incorrect and should be fixed.

9. Scaling of coefficients u_k under L_s\to L_s/\Lambda^2.The claim that rescaling L_s\to L_s/\Lambda^2 corresponds to u_k\to u_k\Lambda^{d_s-2k} is stated without derivation and again uses d_s ambiguously. A short derivation would be helpful, or the formula should be reconsidered.

10. Interpretation of "curvature".The manuscript calls the coefficient u_1(i, i) "scalar curvature". In the continuum, u_1(x,x) is proportional to the scalar curvature only under strong assumptions (Laplace–Beltrami acting on functions, smooth manifold, etc.). On a finite simplicial complex with combinatorial Laplacians, this is heuristic. The paper should be more explicit that this is by analogy with the continuum expansion, not a rigorous discrete scalar curvature derived from a Bochner-type identity.

Numerical and case-study aspects

1. Details of numerical fitting. For the curvature computation, the authors fit the numerically computed K(i, i,t) for t\in(0,1/(4\pi)) to the truncated expansion. The paper would benefit from:
* the number of time points used and the fitting method adopted;
* an indication of goodness of fit or robustness versus the fit window;
* clarification of how they separate contributions from different simplicial orders, given that L_s is block-diagonal.

2. Distance matrices and " roles" of notes.

The qualitative interpretations of the Connes-distance heatmaps (Figs. 7–10, 15–25) are interesting, but they remain primarily descriptive. It would strengthen the paper to add more concrete musical or statistical tests (e.g., cluster analysis, comparisons of tonic/dominant vs. other notes, or correlations with standard music-theory notions).
=========

Clarity, organization, and referencing

1. Section and equation cross-references.
* In Sec. 3.1 The text refers to "section 6 in the Appendix" for details on the spectral density; the Appendix sections are not numbered, and it is hard to know which part is meant.
* At least one reference to "Table 2 in the Supplementary Results section in the Appendix" is correct. Still, the Appendix's organization (especially the "Distance: Constraint Computations" subsection) could be clearer.

1. Figure captions. Some figure captions seem to be partially auto-generated ("FIG. 3: boure", "FIG. 11: minuet", "FIG. 15: fugue 1", etc.), with inconsistent spelling and incomplete descriptions. It would be better to:
* Standardize spellings ("Bourrée", "Minuet I/II", "Fugue from Sonata No. 3", etc.);
* Explain what the axes represent (note names vs. order of first occurrence, as stated in the text);
* Mention the maximum distance values in the captions instead of only in the main text.

2. Reference formatting.
* Some references are incomplete, e.g., [24] lists only "Forman" without initials or full name, and lacks page numbers for the cited paper.
* Several references are very recent arXiv preprints dated 2025; it would be helpful to include arXiv identifiers consistently and to verify that the cited papers (e.g., [15], [38], [41], [43]) are indeed available.

=======

Language, typos, and style

The manuscript would benefit from a thorough language edit. I do not want to list here all issues I have found, but you can find below is a non-exhaustive list of concrete issues, grouped roughly by section.

Abstract and introduction
* “reveal latent geometric structures of these compositions” → “of these compositions”.
* “allows to go beyond the pairwise node-to-node representation” → “allows us to go beyond …” or “allows one to go beyond …”.
* “The latter, which are in this framework higher-order edges” – “The latter” is ambiguous; consider rephrasing to say “These higher-order interactions explicitly…”.
* “therein” is used somewhat archaically; consider replacing it with “in these structures” for clarity.

Higher-order networks and Laplacians (Sec. 2)

* “A Higher-Order Network N” → “A higher-order network N”
* “Boundary operator and Incidence matrices” → either both lower-case or consistently capitalized, but not mixed.
* Several instances of extra spaces around hyphens: “n- dimensional simplex”, “(n–1)- dimensional simplex” should be “n-dimensional”, “(n-1)-dimensional”.
* “tetraherda” (later in Sec. 3.2.1) should be “tetrahedra”.

Geometry section (Sec. 3)

* “Riemanian manifolds” → “Riemannian manifolds”.
* “Spectral Action: the Heat Kernel and Expansion and Scalar Curvature” is slightly awkward; consider “Spectral action, heat kernel expansion, and scalar curvature”.
* “We chose {vi}, {eik}, {fijk}, {tijkl}” → should be “we choose” or “we have chosen”; tense consistency would help.

Case study and conclusions (Secs. 5–6)

* In several places, sentences start with lowercase words or fragments, e.g., “truncating the expansion at t^{4-d_s^{[0]}/2} term” should begin with “Truncating…”.
* “Djikistra algorithm” → “Dijkstra algorithm” (twice, in the Appendix).
* Conclusion has several long sentences that would be clearer if split; for example, the paragraph starting “What does this all mean?” could be broken into shorter sentences, and the random-walk analogy could be more carefully explained.

Miscellaneous
* Throughout, be consistent with “higher-order” vs “higher order”.

Suggestions for improvement

Beyond fixing the specific mathematical and language issues, I suggest:

1. Clarify the mathematical status of the constructions. Distinguish clearly between (i) rigorous results accurate for any finite simplicial complex (e.g., the structure of the Hodge Laplacian and Dirac operator), and (ii) heuristic analogies with smooth Riemannian geometry (e.g., interpreting u_1 as scalar curvature).
2. Strengthen the connection to existing discrete curvature notions. The comparison with Forman–Ricci curvature is a good start. It would be even more convincing to:
* Include at least one quantitative comparison (correlation between node-wise curvatures, or between average curvature and some network statistic);
* Clarify in what sense the proposed curvature captures information that Forman curvature does not.

1. Improve the exposition around the spectral action. A short, self-contained derivation specialized to finite-dimensional operators (no integrals over manifolds) would make the argument more straightforward and avoid mixing continuum notation with the discrete setting.
2. Polish the figures and captions. Since the musical heatmaps are central to the case study, spend a bit more space explaining what they show and how to read them; e.g., mark the tonic and dominant notes on the axes, or add small annotations on the plots.

Conclusion
The paper contains an appealing idea—using spectral triples and the spectral action to define geometric features on higher-order networks—and an attractive musical application. However, the current version has several mathematical inconsistencies (notably around the Connes distance commutator, the use of spectral dimension in the heat-kernel formulas, and the treatment of the (4\pi t)^{-n/2} factor) and numerous language/formatting issues. Addressing these would significantly strengthen the work and make it suitable for publication.

Version 1:

Reviewer comments:

Reviewer #1

(Remarks to the Author)
The authors have solved my previous concerns in the revised version of the manuscript, so it can be accepted for publication.

Reviewer #2

(Remarks to the Author)
The manuscript has greatly improved since the last submission. I still think it is quite technical and hard to follow but I appreciate the efforts by the authors to make it more accessible, self-contained and thorough. As a network and higher-order practitioner, I am still unsure about how or why would I use the proposed method.

Reviewer #3

(Remarks to the Author)
I am satisfied with the improvement made by the authors. Therefore, I suggest the article be accepted for publication.

REPLY TO REFEREES

We thank the Editor for sending us the Referees' reports and for her decision to give our manuscript another chance while expressing clear concerns with quality, which we fully acknowledge. We also thank the Referees for seeing through our mishaps and we are grateful for their in depth review and their care for quality and relevance. We believe their comments and questions helped us situate our work and contribution, clarify the ambiguities, and improve the content and delivery of our work. We trust that our revised manuscript has addressed all their concerns and questions. In the reply below the Referees' questions and concerns are highlighted in teal while changes in the manuscript are highlighted in red.

Reply to the first Referee:

We want to thank the Referee for his/her unwavering care for quality. It guided us tremendously in situating our work and clearly communicating and presenting our contribution.

1. The paper is written in poor English. For example, the abstract begins with: "Our work is concerned with..."

I believe it is more formal to write:

"The present work is concerned with...".

Therefore, a thorough and careful reading and rewriting of the manuscript is necessary. There are also many typographical errors. For instance, the following sentence is unclear:

"Also, inspired by Weyl's law governing the spectrum of the Laplace-Beltrami we operator here we us"

We thank the Reviewer for the thorough reading of the manuscript. The points were addressed and the following changes were made in the manuscript:

The present work is concerned with simplicial complexes that describe higher-order interactions in complex systems.

"Also, inspired by Weyl's law governing the spectrum of the Laplace-Beltrami ~~we~~ operator here we use that of the Hodge Laplacian to infer the dimension of the spaces in which these compositions are embedded."

2. The manuscript is too mathematical and formal, making it difficult to follow. It may not be suitable for the general audience of Communications Physics and would likely be more appropriate for a specialized mathematical-physics journal. Furthermore, some mathematical symbols are not defined, and the authors refer to existing literature for fundamental definitions. A paper of this type should be fully self-contained, presenting all relevant mathematical definitions explicitly so that readers do not need to consult external sources to understand the notation.

We agree with the author that our original submission was not self-contained, which made it hard for the non specialized reader to follow. We trust that this revised version addresses this serious concern that the Reviewer rightfully raised.

3. The main contribution of the present work remains unclear. The authors state in the abstract:

"The prime contribution of this work is the introduction of geometric measures for these simplicial complexes.

However, most of the mathematical derivations presented appear to be already available in the literature, including the geometric measures discussed. Moreover, the application of the theoretical framework does not appear to be new, as the authors have previously published results on closely related topics. The authors must clearly state explicitly in the abstract, introduction, and conclusions what are the original contributions of the present manuscript.

We thank the Referee for helping us refine and clearly state our contribution. We have rewritten the abstract to clarify it:

The present work is concerned with simplicial complexes that describe higher-order interactions in complex systems. This description moves beyond the pairwise representation of simple networks to capture a hierarchy of interactions. Our prime contribution is the introduction of geometric measures for these structures. We achieve this by leveraging the non-commutative algebra of their matrix representations through the application of Connes' spectral triplet formalism. Within this framework, we extend the spectral distance, a metric adapted from Connes' formalism and previously applied to graphs, to higher-order networks, and additionally propose a novel definition of discrete curvature which explicitly depends on the spectral dimension. These serve as characterizing features of higher-order networks and complement known topological metrics. The formalism is demonstrated on a dataset of musical compositions, revealing their latent geometric structures.

4. The quality and explanations of the figures are poor. Many figures contain an excessive number of tick labels, which makes it difficult to clearly visualize the main findings. The

text in the figures is also too small to be easily readable.

We have addressed the Referee's concerns. Concerning the readability of the text in Figures of the Appendix, we had opted to make the Figure sizes smaller to save on space, since we wanted to report all of our results. We appreciate the understanding of the Referee concerning this point. In case of another round of review we would appreciate the Referee's input on what figures to keep.

Minor points

1. Please revise the English grammar and the typographical errors throughout the manuscript.

We are submitting a major revision of the manuscript, including the English grammar and the typographical errors that the Reviewer has pointed out.

2. Many figures are not correctly formatted. For example, some appear in a double-column layout with duplicated captions, which is not appropriate for a single-column manuscript format. I recommend that the authors present such figures as two-panel figures with a single caption that merges the content of the current double captions.

We thank the Reviewer for pointing this out. We have properly aligned the figures and merged the captions where the comment applies.

Reply to the Second Referee: We first want to thank the Referee for their thorough reading of the manuscript and for the time they put in thinking through our proposed framework. We appreciate the perspective he/she is brining to our paper. Below we list the points he/she has raised and address them along with the corresponding modification in the text.

MAJOR CONCERNS

- FORMALITY. Formal, unambiguous mathematical definitions and notation of all key concepts, consistent notation, explaining all symbols used etc see examples below
- READABILITY. Improve readability by explaining the significance of the key concepts used and results from the literature. At the moment is not accessible to a generalist audience. It needs to be much more self-contained if you are addressing the audience of this journal.
- ANALYSIS & INTERPRETATION. In the musical case study, the analysis is mostly descriptive and quite superficial - what does it mean? For example, the captions of all but figure 8 are purely descriptive.
- IMPORTANCE. What is the importance/significance of these results? can they be reproduced with other (simpler) methods? What is the key advantage of this approach and why?
- APPLICABILITY. Can this method be applied to other datasets? If yes, show other examples, if not explain the limitations.

We address his/her major concerns listed above in reverse order. We will start with applicability.

Indeed, the measures we are proposing apply to any dataset. Our choice of the simplicial complexes of Bach's compositions is therefore not restrictive. To illustrate the use of curvature in other contexts, we have carried out experiments in a new Section entitled "Ground checks" in which we have produced test cases showing the applicability of our curvature beyond the music data set. The experiments were designed to check the use of curvature in community detection where other definitions of curvatures also succeed, as as well as when they fail.

These examples on applicability allowed us to also answer the question on importance. Our framework provides a measure that can be used depending on context. For example, the case for edge curvature we showcased an example where our definition correctly predicts the communities when the resolution limit is reached in modularity maximizing algorithms, and where other definitions of curvatures also fail. As for distance, we are proposing it as a generalization in the context of higher-order networks and it clearly could highlight patterns that the common definition of distance as a shortest path over graphs fail to detect. Using the distance matrices in our music example, we focused on the mean tonic-dominant, tonic-subdominant, and tonic-leading tone distances and showed that their values are consistent with music theory, where the dominant-tonic relationship represents the strongest harmonic pull in Western tonal music. This now provides revised interpretation of our results. On readability, in this revised version, and thanks to the comments and concerns of all the Referees, we attempted to provide a self-contained document targeting a general audience, which also addressed the concern on formality. We have checked consistency of definitions, symbols, and notations.

EXAMPLES (non-exhaustive)

p2 higher-order network (hypergraph) 'set of vertices V and hyperedges H' that is not enough - do you allow repetition, is the order important, do you admit weights or only the binary case, what is an orientation, how is it (well) defined, are the elements called hyperedges or simplices, what are delta and d in equation 1, what is the range of n in equation 1

The paragraph was fully rewritten:

A simplicial complex $\mathcal{K}$ is a set of simplices, which is closed under inclusion. Simplices are indexed by their order starting with $n = 0$ for vertices, $n = 1$ for edges, $n = 2$ for triangles, $n = 3$ for tetrahera and so on. Inclusion implies that if a simplex $\in \mathcal{K}$, let us give the example of an edge e for illustration, then its nodes, which constitute the subset of e, must be in $\mathcal{K}$ as well. Further, an orientation in a simplicial complex is given by the effect of the boundary operator ∂. More explicitly, take an edge which is an $n = 1$ simplex: $e = [v_i, v_j]$ with $i < j$, its boundary is $\partial_{n=1}(e) = v_j - v_i$. More generally, for an n-simplex $\sigma = [v_0, \ldots, v_n]$,

$$\partial_n(\alpha) = \sum_{i=0}^{n} (-1)^i \, [v_0, \ldots, \widehat{v}_i, \ldots, v_n],$$

where $[v_0, \ldots, \widehat{v}_i, \ldots, v_n]$ denotes the face with v_i removed. For a triangle $\alpha = [v_0, v_1, v_2]$, its boundary is the alternating sum of its edges:

$$\partial_{n=2}(\alpha) = [v_1, v_2] - [v_0, v_2] + [v_0, v_1].$$

Collecting these relations for all the simplices of the complex defines the incidence matrix $\mathbf{B}_n$ with entries in $\{-1, 0, +1\}$. Subsequently, the entries of the incidence matrices are binary values that indicate whether lower-dimensional simplices are included in higher-dimensional ones [1, 2, 3, 4]. $B_{[1]}$ denotes the incidence matrix of the lowest possible order, which encodes the node to edge relations. It has dimensions $N_{[0]} \times N_{[1]}$, where the first is the number of nodes while the second is the number of edges. More generally, $B_{[n]}$ is $N_{[n-1]} \times N_{[n]}$, $N_{[n]}$ is the number of simplices of order n. An illustration on how to compute the incidence matrices of a simplicial complex $\mathcal{K}$ is given in Figure 1.

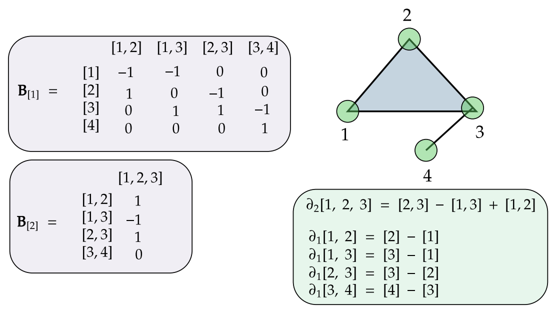


FIGURE 1. Following the orientation convention the matrices $B_{[1]}$ and $B_{[2]}$ of a simplicial complex $\mathcal{K}$ are computed. $\mathcal{K}$ has $N_{[0]} = 4$ nodes, $N_{[1]} = 4$ edges, and $N_{[2]} = 1$ face. This gives $B_1 \in \mathbb{R}^{N_{[0]} \times N_{[1]}} = \mathbb{R}^{4\times 4}$, and $B_2 \in \mathbb{R}^{N_{[1]} \times N_{[2]}} = \mathbb{R}^{4\times 1}$.

p4 missing D, unexplained notation e.g. t_{ijkl}, H_i, functionals (what is the codomain), the norm and bracket $[D, f]$ for graphs and higher-order graphs, equation after eq.3, etc

We noted that we had introduced the degree matrix and also called D, which could be mistakenly understood as the Dirac operator. We made the below changes:

“ The traditional graph Laplacian defined as $L_{[0]} = D_{deg} - A$, where D_{deg} is the diagonal matrix of the nodes' degrees and A is the adjacency matrix. It can be alternatively written as $L_{[0]} = B_{[1]} B_{[1]}^T$, where $B_{[1]}$ is the boundary (incidence) matrix of the lowest order. For a k-simplicial complex, the n-order Hodge Laplacian is represented by the matrix [1, 3]:

$$L_{[n]} = B_{[n]}^T B_{[n]} + B_{[n+1]} B_{[n+1]}^T = L_{[n]}^{\text{down}} + L_{[n]}^{\text{up}}, \tag{1}$$

with $1 \leq n \leq k-1$, and $L_{[k]} = B_{[k]}^T B_{[k]}$. ”

This is a minor modification on the definition of the Dirac operator:

“The square of the Dirac operator D defines the super-Laplacian [5, 6, 7], and acts on the space of all k-simplices for $k = 0, 1, 2, 3 \ldots$. It takes the following block matrix form:”

We expanded on the introduction of the distance and its relation to classical notions:

“In non-commutative geometry Connes introduced a notion of distance throughstates, which are positive linear functionals ϕ and ψ that act on an algebra $\mathcal{A}$ of observables a, and generalize the notion of points in classical spaces. [8, 9, 10]. These functionals are related to the geometric information encoded in the spectrum of D, the Dirac operator, through the following equation that defines the distance d_C:

$$d_C(\phi,\psi) = \sup_{a\in\mathcal{A}} \{|\phi(a) - \psi(a)| : \|[D,a]\| \leq 1\},$$

where $\|[D,a]\|$ is the spectral norm.

Subsequently, the classical notion of measuring distance by finding the shortest path is replaced by a maximization of difference between the states' evaluations of an observable. This maximization is subject to the above constraint which generalizes the classical 1-Lipschitz condition for functions. Recall that classically, a function g, for example defined over $\mathbb{R} \to \mathbb{R}$, is a 1-Lipschitz function when $|g(x) - g(y)| \leq |x - y| \quad$ for all $x, y \in \mathbb{R}$, and the supremum of this inequality $|x - y|$ is nothing but the distance between x and y. In the non-commutative counterpart, the analogue of this is $\|[D,a]\| \leq 1$ since D serves as the non-commutative gradient of a. "

$\mathcal{H}_i$ defines a: "hierarchy of Hilbert spaces $\mathcal{H}_i$" , and $\{t_{ijkl}\}$ are tetrahedra. We made the following modifications to the text:

"As the elementary building blocks of our networks' Hilbert spaces, we choose $\{v_i\}$, $\{e_{ik}\}$, $\{f_{ijk}\}$, $\{t_{ijkl}\}$ corresponding respectively to the vertex, edge, triangle, tetrahedron sets as basis elements of a certain hierarchy of Hilbert spaces $\mathcal{H}_i$ over $\mathbb{C}$ with a scalar product induced by $(v_i|v_k) = \delta_{ik}$ and $(e_{ik}|e_{lm}) = \delta_{il}\delta_{km}$, which carries over to higher-order edges, like triangles and tetrahedra."

p9 how is the simplicial complex constructed? 'C played followed by [D,E,F], an edge is added between C and the 2-simplex [D,E,F]' how? which node (note) do you connect C to in [D,E,F]?

Indeed this is a very important question. A note is connected to its closest voice in the chord [note with the closest frequency], which a musically grounded choice and was explained in our previous work. We agree with the Referee on the need to have the construction procedure explained for completeness. We make the following change to the text for clarification.

"These building blocks are now connected if they occur consecutively, that is if a C is played followed by $[D,E,F]$, an edge is added between C, and the 2-simplex $[D,E,F]$ connecting C to the D in the latter, which is the note with the closest voice (closest frequency)."

p13 the conclusion section mentions things not really investigated: 'performing a random walk'; 'geometric embedding'; 'adapted to these contexts

We thank the Referee for raising these points, which gives us an opportunity to clarify these statements. We made the below changes to the text"

"Although we do not explicitly perform random walks, we were effectively exploring the full graph through the kernel of the heat equation, which describes a continuous-time random walk on the graph. It represents the probability that of reaching a node j after time t, having started at node i. "

Further geometric embedding is by definition giving coordinates to the object under study. We effectively did this by providing a notion of distance between points. The below changes were made:

"They bring in geometric identifiers and measures to supplement the topological ones setting the ground for a Gauss-Bonnet-like relations. The proposed definition of distance provides a geometric embedding, by introducing distances between points, which is a highly active field in network science"

p18 comparison to pairwise distances - normalised as different ranges otherwise misleading e.g. fig24 vs fig27

The Referee is right in pointing out that the ranges are different. We wanted to keep the ranges fixed to compare qualitative differences, which the heatmap can reveal irrespective of scale.

Reply to the third Referee

We thank the Referee for their thorough reading of the manuscript and to their comments and questions, which we highly appreciate. We are grateful for the thinking process that was put into our work to distill and clarify our contribution. We acknowledge that the suggested modifications strengthen our manuscript and remove all the ambiguities that our original submission was flawed with. We go over the points raised by the Referee and list all the associated modifications.

1. Notation and normalization in the density formula. The spectral density for $0 < n < k$ is written as

$$\rho(\lambda) = \frac{N^{[n-1]}}{N^{[n]}} C^{[n-1]} \lambda^{d_s^{[n-1]}/2} + \frac{N^{[n]}}{N^{[n]}} C^{[n]} \lambda^{d_s^{[n]}/2}$$

The second factor $N^{[n]}/N^{[n]}$ is identically 1 and seems to be a typo; presumably the denominator should be the number of simplices of a different order. Furthermore, later, the text says where N is the number of non-zero eigenvalues and N is the number of simplices of the same order, using the same symbol for two different quantities. This must be clarified, and the expression checked carefully.

We agree with the Referee that the notation we have adopted was confusing we make the following changes:

$$\rho(\lambda) = \frac{\mathcal{M}_{[n-1]}}{N_{[n]}} C_{[n-1]} \lambda^{d_S^{[n-1]}/2} + \frac{\mathcal{M}_{[n]}}{N_{[n]}} C_{[n]} \lambda^{d_S^{[n]}/2},$$

"where $\mathcal{M}$ is the number of non-zero eigenvalues of the Laplacian of a given order and N is the number of simplices of the same order".

2. Continuum vs discrete scaling. The derivation of spectral dimensions from the cumulative eigenvalue density should specify the fitting range in λ and the procedure (e.g., log-og fit) used on finite complexes. Right now, the text directly imports the continuum Weyl law and asserts proportionalities without clearly explaining how they are realized in the finite, discrete case.

Indeed, as the Reviewer correctly points out the procedure of fitting was not explained. Below are the changes we made to the text to clarify the methodology:

"The spectral dimension $d_s^{[0]}$ and $d_s^{[k-1]}$, can thus be estimated from the log-log relationship between the eigenvalue λ and the spectral density $\rho(\lambda)$. This can be accomplished by identifying the longest initial range of eigenvalues that exhibits a stable power-law scaling. A linear model of the form $\log \rho \sim \log \lambda$ should be fitted to progressively longer initial subsequences of the ordered eigenvalues. The quality of each fit is evaluated using the Adjusted-R^2. The range that yields the maximal value of the latter is selected as optimal, and its corresponding slope provides an estimate for the respective spectral dimension. The procedure is carried sequently to higher-order, starting with 1, which relies on the identified $d_s^{[0]}$, all the way up to $k-2$, which relies on $d_s^{[k-1]}$."

3. Connes distance and commutator constraints (Sec. 3.2.1 and Appendix)

Misprint in the edge commutator formula.Equation (4) in the main text reads $[B_0, f]\alpha(i,j) = (f_i - f_j)^2(\alpha(i) + \alpha(j))$, while the derivation in the Appendix gives $[B_0, f]\alpha(i,j) = \frac{f_i - f_j}{2}(\alpha(i) + \alpha(j))$, and this latter expression is needed to obtain the constraint $\|[D,f]\| = \|(f_i - f_j)/2\| \leq 1$. So either Eq. (4) or the Appendix must be corrected; as written, they are inconsistent, and Eq. (5) in the main text appears to rely on the Appendix form. This is a genuine mathematical error, not just a typo.

We respectfully want to point out to the Referee that $[B_0, f]\alpha(i,j) = (f_i - f_j)^2(\alpha(i) + \alpha(j))$ is not what we have in the original manuscript. Our equation 4 is consistent with the Equation in the Appendix; it carries no quadratic dependence on $(f_i - f_j)$ but rather a ratio between the latter and 2. In the original submission derivation however, we have used $B_{[0]}$ as our lowest order incidence, which is inconsistent with the indexing which starts at $n = 1$ in the definition of Dirac operator. The calculations remain unchanged. We just update the subscripts for consistency.

"$f \in \Omega^0$ is 0-form (scalar) and $\alpha \in \Omega^1$ is 1-form (co-vector or in simple terms a row vector as opposed to a column vector). Their multiplication should give $\Omega^1 \times \Omega_0 \to \Omega^1$, which means that f has to be pulled up to the dimension of α through an averaging given by:

$$(f\alpha)([i,j]) = (F_0\alpha)([i,j]) = \frac{f_i + f_j}{2}\alpha([i,j]),$$

which is used to find the commutator. This gives:

$$B_1(F_1\alpha(i,j)) = \frac{(f_i + f_j)}{2}(\alpha(j) - \alpha(i)),$$

and

$$F_0(B_1\alpha(i,j)) = F_0(\alpha(j) - \alpha(i)) = f_j\alpha(j) - f_i\alpha(i).$$

Then the commutator is:

$$[D,f]\alpha(i,j) = \frac{(f_i + f_j)}{2}(\alpha(j) - \alpha(i)) - (f_j\alpha(j) - f_i\alpha(i))$$

$$= \frac{f_i}{2}\alpha(j) - \frac{f_i}{2}\alpha(i) + \frac{f_j}{2}\alpha(j) - \frac{f_j}{2}\alpha(i) - f_j\alpha(j) + f_i\alpha(i)$$

$$[D,f]\alpha(i,j) = \frac{(f_i - f_j)}{2}(\alpha(i) + \alpha(j))$$

"

4. Higher-order constraints derived from triangles/tetrahedra.

The formulas (6) and (7) for triangles and tetrahedra follow from the chosen averaging rules, but then the argument that summing over the simplex yields zero and hence $\|[B_1, f]\| = 0$ and $\|[B_2, f]\| = 0$ is too quick. Expand and clarify, please.

We thank the Reviewer for helping us in producing a self-contained document. The mentioned commutators are zero for the case of a graph where f is only defined on nodes. We have substantially expanded this Section of the manuscript and added derivations in the Appendix in support of all the associated derivations. The changes are given below:

"In case we extend the definition of f to edges, triangles, and higher-order simplices, through averaging, the optimization will differ from that on $\mathcal{G}$, of equation **??**, by additional higher-order constraints which will be enumerated shortly.

As given in [11], we start by noting that on graphs f defined solely on vertices:

$$f_{\mathcal{G}} = \begin{bmatrix} F_0 & 0 \\ 0 & 0 \end{bmatrix}, F_0 = \begin{pmatrix} f_1 & 0 & \cdots & 0 \\ 0 & f_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_{N_{[0]}} \end{pmatrix}_{N_{[0]} \times N_{[0]}} ,$$

where F_0 is the block diagonal matrix on vertices. This leads to the commutator on the graph given by:

$$[D, f]_{\mathcal{G}} = \begin{pmatrix} 0 & -F_0 B_1 \\ B_1^{\top} F_0 & 0 \end{pmatrix}$$

which reduces to:

$$\|[D, f]\| = \|[B, f]\| = \|(f_i - f_j)\| \leq 1.$$

Alternatively, if we extend $f = f_{\mathcal{N}}$ by averaging with $F_n = \frac{1}{n+1} \sum_{v \in \sigma} f(v)$ for n-simplex σ, the commutator changes and subsequently the constraints associated with the optimization. We illustrate it below for the case of a simplex of order 1, with f extending over the edges where the block matrix F_1 of size $N_{[1]} \times N_{[1]}$ is defined:

$$f_{\mathcal{N}} = \begin{bmatrix} F_0 & 0 \\ 0 & F_1 \end{bmatrix}$$

Subsequently, the commutator can be written as:

$$[D, f]_{\mathcal{N}} = \begin{bmatrix} 0 & B_1 F_1 - F_0 B_1 \\ B_1^{\top} F_0 - F_1 B_1^{\top} & 0 \end{bmatrix}$$

Clearly, the commutator is now given:

$$[B_1, f]\alpha(i, j) = \frac{(f_i - f_j)}{2}(\alpha(i) + \alpha(j)) \tag{2}$$

We note that equation 2 allows us to the write the constraint $\|[D,f]\| \le 1$ as:

$$\|[D,f]\| = \|\frac{(f_i - f_j)}{2}\| \le 1. \tag{3}$$

The details of this calculation are given in Section 2 in the Appendix along with the higher-order generalizations. Therefore in order to compute the distance between two nodes in the non-commutative space associated with a simplicial complex of order 3, for example, the above optimization problem given in Equation 5 has to be solved with the constraints given in Equation 3, as well as the constraints resulting from the higher-order contributions, given by Equations 5 and 6 of the Appendix in Section 2. More generally, for an arbitrary order n the constraints are then given by Equation 4 below:

$$\|B_{[n]}F_{[n]} - F_{[n-1]}B_{[n]}\| \le 1 \tag{4}$$

The commutator $[D,f]$ contains the block $fB_1 - B_1 f = [B_1, f]$. The calculation above shows that $[B_1, f] = 0$; therefore the contribution of the triangle-edge block to the norm $\|[D,F]\|$ vanishes.

The below changes were made to the Appendix in support of the above:

"The commutator is then given by:

$$[D,f] = \begin{pmatrix} 0 & B_1F_1 - F_0B_1 & 0 & 0 \\ B_1^\top F_0 - F_1 B_1^\top & 0 & B_2F_2 - F_1B_2 & 0 \\ 0 & B_2^\top F_1 - F_2 B_2^\top & 0 & B_3F_3 - F_2B_3 \\ 0 & 0 & B_3^\top F_2 - F_3 B_3^\top & 0 \end{pmatrix}$$

In this case, the boundary operator $B_2 : \Omega^2 \to \Omega^1$ acts on the triangle:

$$B_2(\omega)([i,j,k]) = \omega([j,k]) - \omega([i,k]) + \omega([i,j]).$$

And,

$$\begin{aligned} f(B_2\omega)([i,j,k]) &= f\big(\omega([j,k]) - \omega([i,k]) + \omega([i,j])\big) = F_1\big(\omega([j,k]) - \omega([i,k]) + \omega([i,j])\big) \\ &= \frac{f_j + f_k}{2}\omega([j,k]) - \frac{f_i + f_k}{2}\omega([i,k]) + \frac{f_i + f_j}{2}\omega([i,j]). \end{aligned}$$

$$B_2(f\omega)([i,j,k]) = B_2(F_2\omega)([i,j,k]) = \frac{f_i + f_j + f_k}{3} B_2\omega([i,j,k]) = \frac{f_i + f_j + f_k}{3}[\omega([j,k]) - \omega([i,k]) + \omega([i,j])$$

Then the commutator is given by:

$$[B_2, f]\omega([i,j,k]) = \frac{f_j + f_k - 2f_i}{6}\omega([j,k]) + \frac{-f_i + 2f_j - f_k}{6}\omega([i,k]) + \frac{f_i + f_j - 2f_k}{6}\omega([i,j]). \tag{5}$$

The commutator $[D, f]$ contains the block $fB_2 - B_2 f = [B_2, f]$ and $fB_3 - B_3 f = [B_2, f]$. The calculation above shows that they are both $\neq 0$ only when we extend f to higher-order simplices; otherwise their to the norm $\|[D, F]\|$ vanish.

Finally, for an oriented tetrahedron $[i, j, k, l]$, f has to be pulled to the dimension of $\beta \in \Omega^3$ through an averaging given by:

$$(f\beta)([i, j, k, l]) = (F_3\beta)([i, j, k, l]) = \frac{f_i + f_j + f_k + f_l}{4}\beta([i, j, k, l]),$$

The corresponding entry in the commutator is:

$$\begin{aligned}(B_3F_3 - F_2B_3)\beta([i, j, k, l]) = \frac{1}{12}\Big[&(3f_l - f_i - f_j - f_k)\,[i, j, k] \\ &-(3f_k - f_i - f_j - f_l)\,[i, j, l] \\ &+(3f_j - f_i - f_k - f_l)\,[i, k, l] \\ &-(3f_i - f_j - f_k - f_l)\,[j, k, l]\Big]\,\beta([i, j, k, l]).\text{”}\end{aligned} \tag{6}$$

The step Therefore, we enforce the constraint on all pairwise interactions in the simplex $\|(f_i - f_j)/2\| \leq 1, \ldots$ is not mathematically justified; it is a design choice. This must be clearly stated as an assumption rather than as a consequence of the commutator structure.

We acknowledge that we have introduced a confusion in this part of the original manuscript. This applies to the case where the simplex has f defined over the edges and the nodes only, and not on higher order simplices. We hope that the modifications listed above clarify this point.

5. Norm used in $\|[D, f]\|$. The operator norm used in the constraint $\|[D, f]\| \leq 1$ should be specified precisely (spectral norm? Hilbert-Schmidt?). This matters when passing from matrix inequalities to scalar inequalities on individual edge differences.

Indeed the Referee is right in pointing this out. We made the below changes:

“where $\|[D, a]\|$ is the spectral norm. ”

6. Generalization of the Connes distance formula.The definition $\mathrm{dist}_{C,N}(i, j) = \sup\{|f_j - f_i| : \|[D, f]\| \leq 1\}$ is imported from graphs, but now the Dirac operator acts on a direct sum over 0-, 1-, 2- and 3-simplices. The authors restrict attention to constraints on edges only; this should be argued more carefully (e.g., by showing that higher-order blocks do not change the supremum, or by explaining why it is reasonable to ignore them).

We hope that the clarifications provided above answered all the questions concerning this part.

Spectral action, heat kernel and curvature (Sec. 3.2.2) Confusing and probably incorrect use of d_s, D, and Λ. After introducing the spectral action $\operatorname{Tr} F(D/\Lambda)$, the manuscript stateswhere $\{\lambda_i\}$ are the eigenvalues of D, and $d_s = |D^{-1}| = |1/\Lambda_1| = 1/\Lambda$ Here D^{-1} is an operator, while d_s was previously the spectral dimension vector. Identifying a dimension with the inverse of the smallest eigenvalue is dimensionally and conceptually incorrect. The sentence should be removed or rewritten entirely; at the moment, it mixes three unrelated notions: spectral dimension, inverse Dirac operator, and cutoff Λ.

We respectfully want to point out that this is not d_s the spectral curvature but rather ds the line element: $ds = |D^{-1}| = |1/\Lambda_1|$. This is not of relevance to our context. We removed it in this revised version it as it does not have any implications in what followed it.

7. Mismatch of exponents and dimensions in Eq. (12). The diagonal heat kernel expansions are written as (for vertices)$K_v(i,i,t) \approx u_{0,v}(i,i)(4\pi t)^{-d_s^{[0]}/2} + u_{1,v}(i,i)(4\pi t)^{-d_s^{[0]}/2} t + O(t^{2-d_s^{[0]}/2})$. But in the continuum, the $(4\pi t)^{-n/2}$ prefactor involves the topological (manifold) dimension n, not the spectral dimension. If the authors want to replace n by the empirically fitted $d_s^{[0]}$, they should justify this substitution; otherwise, the scaling in (12) is not derived but assumed. The same applies to the edge and face formulas

Indeed. There Referee is correct in pointing this out. We define it by analogy and show that it is a reasonable choice with the below empirical evidence.

1. Ground Checks

In this section, we perform controlled experiments using the Stochastic Block Model, to generate networks with varying size, community structure, and intra/inter-community connection probabilities p_{intra} and p_{inter}. The process starts with the adjacency and degree matrices to build the network Laplacian. The spectrum of the latter is computed and the spectral dimension d_s is recovered from Equation **??**. The framework is summarized in Figure 2.

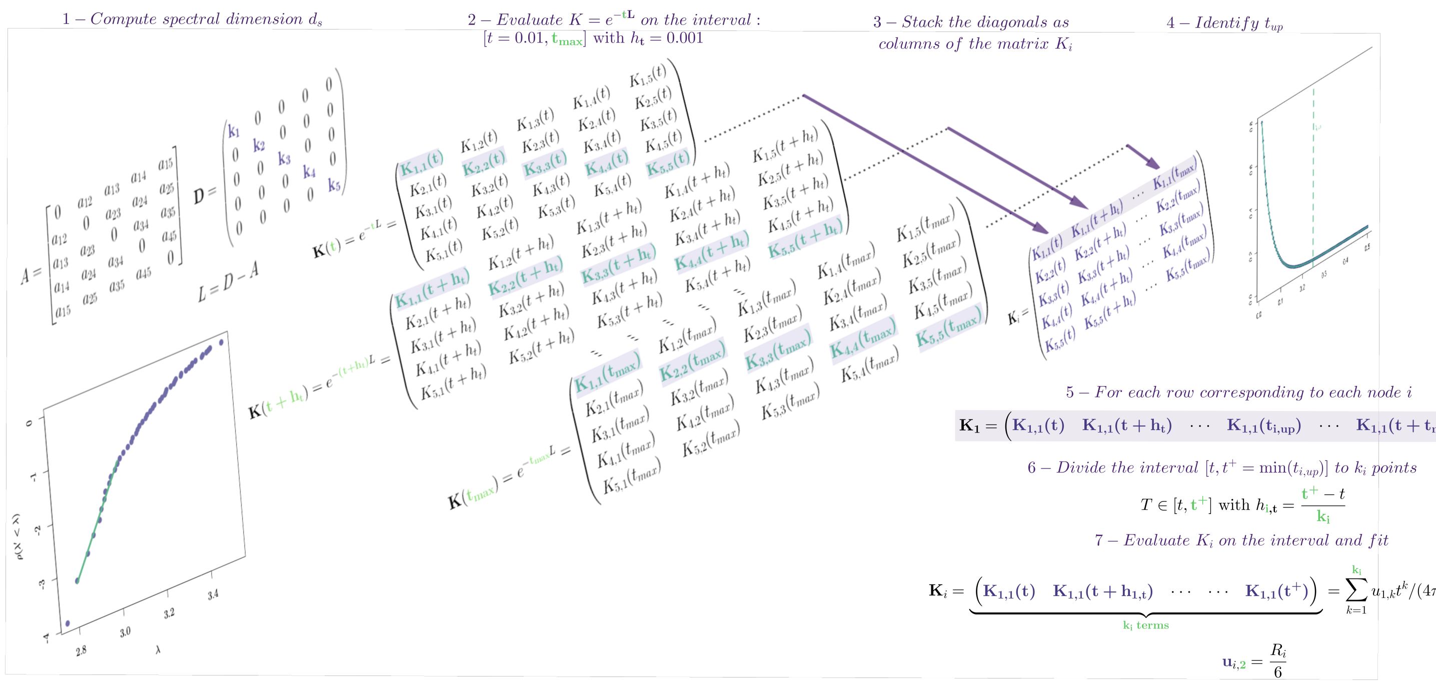
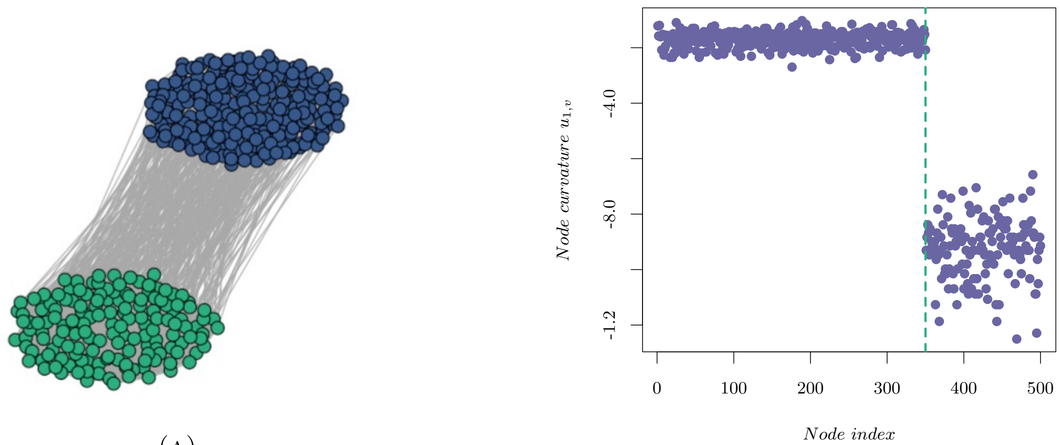


Figure 2. The heat kenel expansion workflow.

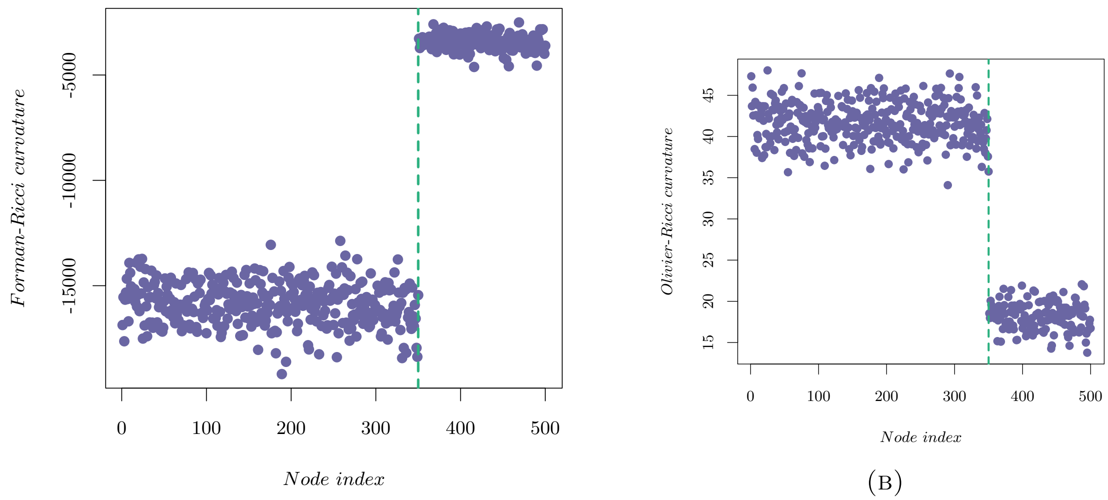
Below, we show the result of one such realization in which the graph's nodes' curvatures are computed following our formalism. The graph is shown in Figure 3a, while Figure 3b shows the clear separation of the nodes into two communities based on their curvatures, accurately predicting their community membership.



(A) (B)

FIGURE 3. SBM with $N = 500$, $p_{\text{inter}} = 0.01$, $p_{\text{intra}} = 0.5$, with two communities of sizes 350 and 150.

This provided us with empirical evidence that identifying q, the manifold dimension in the continuum, with d_s in Equation **??** is justified since it yielded a measure which correctly separated the network into its communities. We further carried out the computation of other measures of curvature, particularly Forman-Ricci and Olivier using GraphRicciCurvature in Python, for comparison which are shown in Figures 4a and 4b respectively. Their respective correlations with $u_{1,v}$ are -0.985 and 0.987. We carried out many similar experiments giving us confidence in our construction, which originally relied on an analogy that we drew with the continuum. We thus stress that this is empirically supported and we do not claim to have a proof for its validity.

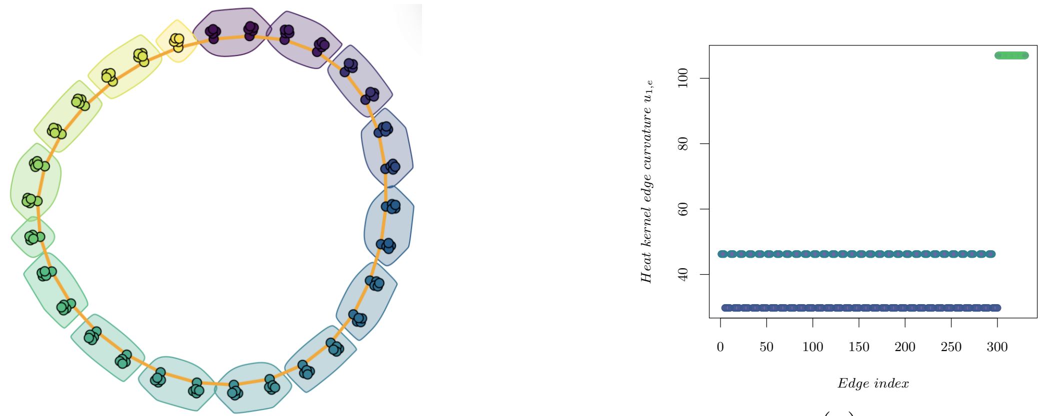


(A)

(B)

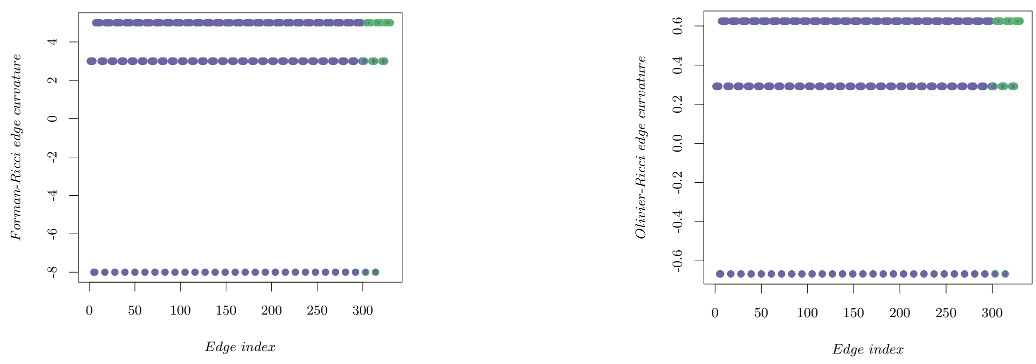
FIGURE 4. The Forman and Olivier-Ricci curvature of the graph in Figure 3.

We further tested our method on edge classification. We constructed a ring of cliques shown in Figure 5a, where the edges connecting the nodes on the ring are highlighted in orange, and the convex hulls are the identified communities using Louvain algorithm [12]. Note that the cliques are the correct communities in this graph and Louvain fails to identify them. We put our method to test and computed the edge curvatures $u_{1,e}$, and compared them to Forman and Olivier-Ricci curvatures, which are commonly used in edge classification and subsequently community detection. The edges connecting the ring nodes are clearly identified with $u_{1,e}$, whose curvatures values are highlighted in green in Figure 5b, while Forman and Olivier-Ricci curvatures failed to classify them as shown in Figures 6a and 6b respectively. The dark green corresponds to the edges joining the ring nodes to the cliques members, and the dark blue are the edges in the cliques excluding the ring nodes.



(A) (B)

FIGURE 5. A graph with 30 cliques connected by a ring is shown in the 5a, with a cliques size of 5. For this graph $u_{1,e}$ are plotted in 5b, which clearly distinguish the orange edges. Their curvature values are shown in green distinguishing them from the other edges of the graph. The others are the edges connecting the nodes of the clique to the node on the ring, and the nodes of the cliques among each other excluding the ring node.



(A) (B)

FIGURE 6. Both Forman and Olivier-Ricci edge curvatures are shown in 6a and 6b respectively, which clearly fail to distinguish the orange edges of the graph, whose curvature values are highlighted in green.

This provides further evidence for the importance of our method especially where the modularity maximization methods, such as Louvain, wrongly lumps the nodes into the convex hulls shown in 5a.

We showed this example because it is one that highlights the resolution limit in community detection [13], and an example in which $u_{1,e}$ carried sufficient information to distinguish the communities. In particular, the removal of small fraction of edges, those which correspond to $u_{1,e} > 100$, leads to the recovery of the correct communities. This procedure is similar to the Ricci flow surgery, in which the deformation of the metric of Riemannian manifold g_{ij} is driven by curvature. The metric evolves under the effect of the flow given by $\frac{\partial g_{ij}}{\partial t} = -R_{ij}$, with the latter being scalar curvature. Subsequently positively curved regions in the manifold tend to shrink while negatively curved ones undergo expansion. This formalism was adapted in the network setting and is the discrete equivalent of the Ricci flow [14]. Edge weights are updated following $w_{ij}(t+1) = \text{dist}_{C,\mathcal{G}}(i,j) - \kappa_{ij}\text{dist}_{C,\mathcal{G}}(i,j)$, where κ_{ij} is Forman/Olivier-Ricci curvature. Similarly, highly positive curved edges tend to shrink revealing communities. Here we modify it by replacing $\text{dist}_{C,\mathcal{G}}(i,j)$ with Connes's distance and κ_{ij} with $u_{1,e}$. Subsequently, Equation 7 is our proposed modified Ricci-flow:

$$\frac{\partial w_{ij}}{\partial t} = -u_{1,e_{ij}} w_{ij} \tag{7}$$

$w_{ij}(t=0) = \text{dist}_{C,\mathcal{N}}(i,j)$ then carry out the procedure of surgery given in [14].

8. Inconsistent treatment of the $(4\pi t)^{-n/2}$ factor. To reconcile the exact series (14) and the heat-kernel expansion (15), the authors writewe take $(4\pi t)^{-n/2} \to 1$, which now defines $t^+ \to 0^+$ Mathematically, as $t \to 0^+$, $(4\pi t)^{-n/2}$ diverges; it cannot be approximated by 1. One could factor it out and absorb it into the coefficients, or work with a fixed small but finite t, but the present statement is incorrect and should be fixed.

This is a crucial point. Indeed, we expand on the computation associated with t^+ and clarify its details.

"Similar to the continuum counterpart of Equation 12, we let $x \to i$, and focus on the diagonals of the kernel $K(i,i,t)$. We further replace q, the dimension of the manifold, with the associated spectral dimension d_s. Due to the block structure of L_s given in Equation 2, $K(i,i,t)$ is defined on nodes, edges, and triangles. Thus, it can be split into their corresponding components: $K_v(i,i,t)$ with $N_{[0]}$ elements, $K_e(i,i,t)$ with $N_{[1]}$ elements, and $K_f(i,i,t)$ with $N_{[2]}$ elements. This translates into:

$$K(i,i,t) \sim \frac{1}{(4\pi t)^{d_s/2}} \sum_{m=0}^{m_i^{[n]}} u_m(i,i) t^m, \tag{8}$$

where the upper limit in the expansion $m_i^{[n]}$ runs over to the generalized degree of the simplex i of order n [1, **?**], to avoid over/under-fitting. For example, for $n = 0$ and $i \in [1; N_{[0]}]$, $m_i^{[0]}$ denotes the degree of node i; for $n = 1$, $i \in [N_{[0]} + 1; N_{[1]}]$, $m_i^{[1]}$ denotes the generalized degree of edge i. The argument is explained in Section 3.1 in the Appendix. This carries over all orders of the simplex. Below we write the expansions of the vertices

v, edges e, and triangular faces f:

$$K_v(i,i,t) \approx u_{0,v}(i,i)\left(\frac{1}{4\pi t}\right)^{d_s^{[0]}/2} + u_{1,v}(i,i)\left(\frac{1}{4\pi t}\right)^{d_s^{[0]}/2} t + \cdots + \mathcal{O}(t^{m_i^{[0]}+1-d_s^{[0]}/2}), \tag{9}$$

Likewise:

$$K_e(i,i,t) \approx u_{0,e}(i,i)\left(\frac{1}{4\pi t}\right)^{d_s^{[1]}/2} + u_{1,e}(i,i)\left(\frac{1}{4\pi t}\right)^{d_s^{[1]}/2} t + \cdots + \mathcal{O}(t^{m_i^{[1]}+1-d_s^{[1]}/2}),$$

$$K_f(i,i,t) \approx u_{0,f}(i,i)\left(\frac{1}{4\pi t}\right)^{d_s^{[2]}/2} + u_{1,f}(i,i)\left(\frac{1}{4\pi t}\right)^{d_s^{[2]}/2} t + \cdots + \mathcal{O}(t^{m_i^{[2]}+1-d_s^{[2]}/2}),$$

"

"Therefore, in order to determine t^+, $K(i,i,t)$ should be evaluated over an original $T \in [t = 1/\lambda_{max}, t_{max} \leq 1]$, where λ_{max} is the largest eigenvalue of $L_{[n]}$. Then using a segmented regression approach the first point of change in behavior $\log K(i,i,t)$ versus T is identified $t_{i,up}$. Subsequently, t^+ is defined as the minimum over the $t_{i,up}$. Having identified the vicinity of 0, the kernel is then evaluated over $[1/\lambda_{max}, t^+]$ for each node dividing the interval into m_i points. This ensures having enough data points to retrieve the m coefficients u_m and to avoid over/under-fitting."

9. Scaling of coefficients u_k under $L_s \to L_s/\Lambda^2$.The claim that rescaling $L_s \to L_s/\Lambda^2$ corresponds to $u_k \to u_k\Lambda^{d_s-2k}$ is stated without derivation and again uses d_s ambiguously. A short derivation would be helpful, or the formula should be reconsidered.

"If $L'_s = L_s/\Lambda^2$, then the transformation $L_s \to L_s/\Lambda^2$ is physically equivalent to the transformation $t \to t\Lambda^2$ leading to:

$$K(i,i,t) \sim \frac{1}{(4\pi t\Lambda^2)^{d_s/2}} \sum_{m=0}^{\infty} u_m(x,y)\Lambda^{2m}t^m,$$
$$K(i,i,t) \sim \frac{1}{(4\pi t)^{d_s/2}} \sum_{k=0}^{\infty} u_m(x,y)\Lambda^{2m-d_s}t^m,$$

"

For this to look like a standard heat kernel expansion for L' with new coefficients u'_k, we must define:

$$u'_k(x) \equiv u_k(x) \cdot \Lambda^{2k-d}$$

10. Interpretation of curvature.The manuscript calls the coefficient $u_1(i,i)$ scalar curvature. In the continuum, $u_1(x,x)$ is proportional to the scalar curvature only under strong

assumptions (LaplaceBeltrami acting on functions, smooth manifold, etc.). On a finite simplicial complex with combinatorial Laplacians, this is heuristic. The paper should be more explicit that this is by analogy with the continuum expansion, not a rigorous discrete scalar curvature derived from a Bochner-type identity.

Indeed it is analogy and we hope that the cases studies provided empirical support for its validity. We made the below changes in the text to clarify it:

"This provided us with empirical evidence that identifying q, the manifold dimension in the continuum, with d_s in Equation 11is justified since it yielded a measure which correctly separated the network into its communities. We further carried out the computation of other measures of curvature, particularly Forman-Ricci and Olivier using GraphRicciCurvature in Python, for comparison which are shown in Figures 4a and 4b respectively. Their respective correlations with $u_{1,v}$ are -0.985 and 0.987. We carried out many similar experiments giving us confidence in our construction, which originally relied on an analogy that we drew with the continuum. We thus stress that this is empirically supported and we do not claim to have a proof for its validity."

Numerical and case-study aspects

1. Details of numerical fitting. For the curvature computation, the authors fit the numerically computed $K(i,i,t)$ for $t \in (0, 1/(4\pi))$ to the truncated expansion. The paper would benefit from:

* the number of time points used and the fitting method adopted;

* an indication of goodness of fit or robustness versus the fit window;

* clarification of how they separate contributions from different simplicial orders, given that L_s is block-diagonal.

All the aspects of the numerical fitting were explained in answering point 8 raised by the Referee.

2. Distance matrices and functional roles of notes.

The qualitative interpretations of the Connes-distance heatmaps (Figs. 710, 1525) are interesting, but they remain primarily descriptive. It would strengthen the paper to add more concrete musical or statistical tests (e.g., cluster analysis, comparisons of tonic/dominant vs. other notes, or correlations with standard music-theory notions).

We thank the Referee for this very important suggestion. The associated changes are given below:

"We computed the mean tonic-dominant, tonic-subdominant, and tonic-leading tone mean distances, which were given respectively by 2 ± 0, 2.78 ± 1.00, and 2.89 ± 1.03. This is consistent with music theory, where the dominant-tonic relationship represents the strongest harmonic pull in Western tonal music. The effect size was substantial (Cohen's $d = 0.882$) indicating that Connes's distance provides a statistically significant geometric separation of the notes' harmonic functions. This applied to all the compositions."

Clarity, organization, and referencing

1. Section and equation cross-references.

* In Sec. 3.1 The text refers to section 6 in the Appendix for details on the spectral density; the Appendix sections are not numbered, and it is hard to know which part is meant.

We thank the Referee for pointing this out. Indeed, it was hard to follow without proper numbering of the sections in the Appendix. We have now numbered them to make the transitions between the main text in the Appendix smoother.

* At least one reference to Table 2 in the Supplementary Results section in the Appendix is correct. Still, the Appendix's organization (especially the Distance: Constraint Computations subsection) could be clearer.

We numbered sections in the Appendix for clarity.

1. Figure captions. Some figure captions seem to be partially auto-generated (FIG. 3: boure, FIG. 11: minuet, FIG. 15: fugue 1, etc.), with inconsistent spelling and incomplete descriptions. It would be better to:

* Standardize spellings (Bourre, Minuet I/II, Fugue from Sonata No. 3, etc.);

* Explain what the axes represent (note names vs. order of first occurrence, as stated in the text);

* Mention the maximum distance values in the captions instead of only in the main text.

2. Reference formatting.

* Some references are incomplete, e.g., [24] lists only Forman without initials or full name, and lacks page numbers for the cited paper.

Indeed we were missing the author's first name.

* Several references are very recent arXiv preprints dated 2025; it would be helpful to include arXiv identifiers consistently and to verify that the cited papers (e.g., [15], [38], [41], [43]) are indeed available.

We added the identifiers of these papers as suggested by the Reviewer.

Language, typos, and style

The manuscript would benefit from a thorough language edit. I do not want to list here all issues I have found, but you can find below is a non-exhaustive list of concrete issues, grouped roughly by section.

Abstract and introduction

* reveal latent geometric structures of these compositions → of these compositions.

The statement is removed in this revised version.

* allows to go beyond the pairwise node-to-node representation → allows us to go beyond or allows one to go beyond .

This section has been rewritten. We changed it to:

The present work is concerned with simplicial complexes that describe higher-order interactions in complex systems. This description moves beyond the pairwise representation of simple networks to capture a hierarchy of interactions.

* The latter, which are in this framework higher-order edges The latter is ambiguous; consider rephrasing to say These higher-order interactions explicitly.

The change was made as suggested by the Referee.

* therein is used somewhat archaically; consider replacing it with in these structures for clarity.

The sentence was changed to the following:

The fact that these structures can be studied within an algebraic topological framework raises the question of a link to discrete differential geometry, opening the possibility of revealing latent geometric features.

Higher-order networks and Laplacians (Sec. 2)

* A Higher-Order Network N → A higher-order network N:

This statement was removed in this revised version.

* Boundary operator and Incidence matrices → either both lower-case or consistently capitalized, but not mixed.:

This statement was also removed in this revised version.

Several instances of extra spaces around hyphens: n- dimensional simplex, (n1)- dimensional simplex should be n-dimensional, (n-1)-dimensional.

* tetraherda (later in Sec. 3.2.1) should be tetrahedra.

Geometry section (Sec. 3)

* Riemanian manifolds → Riemannian manifolds.

* Spectral Action: the Heat Kernel and Expansion and Scalar Curvature is slightly awkward; consider Spectral action, heat kernel expansion, and scalar curvature.

* We chose vi, eik, fijk, tijkl → should be we choose or we have chosen; tense consistency would help.

We thank the Referee for reading the paper with patience. All the above were addressed.

Case study and conclusions (Secs. 56)

* In several places, sentences start with lowercase words or fragments, e.g., truncating the expansion at $t^{4-d_s^{[0]}/2}$ term should begin with Truncating.

* Djikistra algorithm → Dijkstra algorithm (twice, in the Appendix).

All the above were fixed.

* Conclusion has several long sentences that would be clearer if split; for example, the paragraph starting What does this all mean? could be broken into shorter sentences, and the random-walk analogy could be more carefully explained.

Miscellaneous

* Throughout, be consistent with higher-order vs higher order.

We changed all occurrences as suggested by the Referee.

Suggestions for improvement

Beyond fixing the specific mathematical and language issues, I suggest:

1. Clarify the mathematical status of the constructions. Distinguish clearly between (i) rigorous results accurate for any finite simplicial complex (e.g., the structure of the Hodge Laplacian and Dirac operator), and (ii) heuristic analogies with smooth Riemannian geometry (e.g., interpreting u_1 as scalar curvature).

We have addressed this based on the Reviewer's concern and listed it in the changes above.

2. Strengthen the connection to existing discrete curvature notions. The comparison with FormanRicci curvature is a good start. It would be even more convincing to:

* Include at least one quantitative comparison (correlation between node-wise curvatures, or between average curvature and some network statistic);

* Clarify in what sense the proposed curvature captures information that Forman curvature does not.

The newly added section on cases study addresses these important points.

1. Improve the exposition around the spectral action. A short, self-contained derivation specialized to finite-dimensional operators (no integrals over manifolds) would make the argument more straightforward and avoid mixing continuum notation with the discrete setting.

2. Polish the figures and captions. Since the musical heatmaps are central to the case study, spend a bit more space explaining what they show and how to read them; e.g., mark the tonic and dominant notes on the axes, or add small annotations on the plots.

References

[1] G. Bianconi, Higher-order networks, Cambridge University Press, 2021.

[2] A. P. Millán, J. G. Restrepo, J. J. Torres, G. Bianconi, Geometry, topology and simplicial synchronization, in: Higher-Order Systems, Springer, 2022, pp. 269–299.

[3] J. J. Torres, G. Bianconi, Simplicial complexes: higher-order spectral dimension and dynamics, Journal of Physics: Complexity 1 (1) (2020) 015002.

[4] A. P. Millán, H. Sun, L. Giambagli, R. Muolo, T. Carletti, J. J. Torres, F. Radicchi, J. Kurths, G. Bianconi, Topology shapes dynamics of higher-order networks, Nature Physics (2025) 1–9.
[5] G. Bianconi, The topological dirac equation of networks and simplicial complexes, Journal of Physics: Complexity 2 (3) (2021) 035022.
[6] L. Calmon, M. T. Schaub, G. Bianconi, Higher-order signal processing with the dirac operator, in: 2022 56th Asilomar Conference on Signals, Systems, and Computers, IEEE, 2022, pp. 925–929.
[7] L. Calmon, M. T. Schaub, G. Bianconi, Dirac signal processing of higher-order topological signals, New Journal of Physics 25 (9) (2023) 093013.
[8] A. Connes, H. Moscovici, The local index formula in noncommutative geometry, Geometric & Functional Analysis GAFA 5 (2) (1995) 174–243.
[9] A. Connes, J. Cuntz, M. A. Rieffel, Noncommutative geometry, Oberwolfach Reports 6 (3) (2010) 2271–2334.
[10] A. Connes, On the spectral characterization of manifolds, Journal of Noncommutative Geometry 7 (1) (2013) 1–82.
[11] M. Requardt, Dirac operators and the calculation of the connes metric on arbitrary (infinite) graphs, Journal of Physics A: Mathematical and General 35 (3) (2002) 759.
[12] V. D. Blondel, J.-L. Guillaume, R. Lambiotte, E. Lefebvre, Fast unfolding of communities in large networks, Journal of statistical mechanics: theory and experiment 2008 (10) (2008) P10008.
[13] S. Fortunato, M. Barthelemy, Resolution limit in community detection, Proceedings of the national academy of sciences 104 (1) (2007) 36–41.
[14] C.-C. Ni, Y.-Y. Lin, F. Luo, J. Gao, Community detection on networks with ricci flow, Scientific reports 9 (1) (2019) 9984.

We thank the Editor for sending us the Referees reports and we thank the Referees for helping us improve our manuscript's content and delivery and style, which they are all pleased with. Below we provide their comments which require no further reply from our end.

Changes to the Abstract, end of the Introduction section, and figure captions are highlighted in red in this revised version of the manuscript in which we followed the Editorial requirements and guidelines.

REVIEWERS' COMMENTS:

Reviewer #1 (Remarks to the Author):

The authors have solved my previous concerns in the revised version of the manuscript, so it can be accepted for publication.

Reviewer #2 (Remarks to the Author):

The manuscript has greatly improved since the last submission. I still think it is quite technical and hard to follow but I appreciate the efforts by the authors to make it more accessible, self-contained and thorough. As a network and higher-order practitioner, I am still unsure about how or why would I use the proposed method.

Reviewer #3 (Remarks to the Author):

I am satisfied with the improvement made by the authors. Therefore, I suggest the article be accepted for publication.