

Roughening and dynamics of an electric flux string in a (2+1)D lattice gauge theory: Supplementary Information

Francesco Di Marcantonio,^{1,*} Sunny Pradhan,^{1,†} Sofia Vallecorsa,^{2,‡}

Mari Carmen Bañuls,^{3,4,§} and Enrique Rico Ortega^{1,2,5,6,¶}

¹*EHU Quantum Center and Department of Physical Chemistry, University of the Basque Country UPV/EHU, P.O. Box 644, 48080 Bilbao, Spain*

²*European Organization for Nuclear Research (CERN), 1211 Geneva 23, Switzerland*

³*Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Str. 1, D-85748 Garching, Germany*

⁴*Munich Center for Quantum Science and Technology (MCQST), Schellingstr. 4, D-80799 München, Germany*

⁵*Donostia International Physics Center, 20018 Donostia-San Sebastián, Spain*

⁶*IKERBASQUE, Basque Foundation for Science, Plaza Euskadi 5, 48009 Bilbao, Spain*

Supplementary Note A: Duality transformation with static charges

We consider a lattice Λ with L plaquettes in the horizontal direction $\hat{1}$ and N plaquettes in the vertical direction $\hat{2}$. The sites are labeled as $v = (x, y)$, with integer coordinates x and y along $\hat{1}$ and $\hat{2}$, respectively. With periodic boundary conditions in the vertical direction, we have $x = 1, \dots, L+1$ and $y = 1, \dots, N$, with $y + N \equiv y$. From Λ we define the dual lattice Λ^* , where the plaquettes of the former are the sites of the latter. Each plaquette $\square$ (hence dual site) is labeled with its direct site v in the bottom left corner.

The main idea of the duality is to map the plaquette states into spin- $\frac{1}{2}$ states. We first define the set of operators μ_v^x and μ_v^z for each site of the dual lattice. They have the same operator algebra as σ^z and σ^x . Then, we define μ_v^x as the plaquette operator at the site v

$$\mu_v^x = (\sigma^z \sigma^z \sigma^z \sigma^z)_{\square_v}. \quad (\text{A1})$$

Subsequently, we can define μ^z at $v = (x, y)$ as a string of σ^x starting from the left boundary $x = 1$:

$$\mu_{(x,y)}^z = \prod_{j=1}^x \sigma_{(j,y),\hat{2}}^x. \quad (\text{A2})$$

Notice that the definition of μ_v^z involves only the σ^x operators on the vertical links.

To write down the dual Hamiltonian, we need to invert (A1) and (A2) and express the operators σ^z and σ^x as a function of μ^x and μ^z . The plaquette term is given by (A1) by definition, while the electric terms require a bit more work. On the vertical links, away from the boundary, we observe from (A2) that the electric field can be written down as an interaction term:

$$\sigma_{(x,y),\hat{2}}^x = \mu_{(x,y)}^z \mu_{(x-1,y)}^z. \quad (\text{A3})$$

On the leftmost boundary $x = 1$, μ^z and σ_2^x coincide, meaning

$$\mu_{(1,y)}^z = \sigma_{(1,y),\hat{2}}^x. \quad (\text{A4})$$

Now the goal is to find an expression for the electric fields σ_1^x on the horizontal links, which can be achieved by consecutive applications of the Gauss law. We start with the leftmost boundary $x = 1$, by considering the action on physical states of the Gauss law at a site $(1, y)$:

$$G_{(1,y)} = \sigma_{(1,y),\hat{1}}^x \sigma_{(1,y),\hat{2}}^x \sigma_{(1,y),-\hat{2}}^x = q_{(1,y)}. \quad (\text{A5})$$

We can substitute (A4) in place of $\sigma_{(1,y),\hat{2}}^x$ and $\sigma_{(1,y),-\hat{2}}^x$ in the above expression and obtain

$$\sigma_{(1,y),\hat{2}}^x = q_{(1,y)} \mu_{(1,y)}^z \mu_{(1,y-1)}^z. \quad (\text{A6})$$

Notice that $\sigma_{(1,y),\hat{2}}^x$ depends on the charge $q_{(1,y)}$. Then, we move to the right and consider the column $x = 2$. Here, the Gauss law at the site $(2, y)$ reads:

$$G_{(2,y)} = \sigma_{(2,y),\hat{1}}^x \sigma_{(2,y),\hat{2}}^x \sigma_{(2,y),-\hat{1}}^x \sigma_{(2,y),-\hat{2}}^x = q_{(2,y)}. \quad (\text{A7})$$

Now we can substitute both (A3) and (A6) and express $\sigma_{(2,y),\hat{1}}^x$ in terms of dual operators:

$$\sigma_{(2,y)}^x = q_{(1,y)} q_{(2,y)} \mu_{(2,y)}^z \mu_{(2,y-1)}^z. \quad (\text{A8})$$

We see that now it depends on the product of charges $q_{(1,y)} q_{(2,y)}$. This process can then be iterated for all the other fields on the horizontal links, obtaining:

$$\sigma_{(x,y),\hat{1}}^x = \eta(x, y) \mu_{(x,y)}^z \mu_{(x,y-1)}^z, \quad (\text{A9})$$

with a phase factor

$$\eta(x, y) = \prod_{j=1}^x q_{(j,y)} \quad (\text{A10})$$

that depends on a string of charges on the row relative to that horizontal link. We can write down the mapping of the electric fields in a general way as

$$\sigma_{v,\hat{\mu}}^x \mapsto \omega(v, \hat{\nu}) \mu_v^z \mu_{v-\hat{\nu}}^z, \quad (\text{A11})$$

* fra.di.marcantonio@gmail.com

† sunny.pradhan@ehu.eus

‡ sofia.vallecorsa@cern.ch

§ banulsm@mpq.mpg.de

¶ enrique.rico.ortega@gmail.com

where $\hat{v}$ is the direction perpendicular to $\hat{\mu}$. In our case we have ($v = (x, y)$)

$$\omega(v, \hat{v}) = \begin{cases} 1 & \hat{v} = 1, \\ \prod_{j=1}^x q_{(j,y)} & \hat{v} = 2. \end{cases} \quad (\text{A12})$$

This general description for vertical (A3) and horizontal (A9) links is depicted in Suppl. Fig. 1. From (A11) and (A12), we see that the Gauss law is automatically satisfied:

$$G_v = \omega(v, \hat{2})\omega(v, \hat{1})\omega(v - \hat{1}, \hat{2})\omega(v - \hat{2}, \hat{1}) = q_v. \quad (\text{A13})$$

Eventually, we deal with the rightmost boundary of vertical links. We can actually assign N additional dual sites relative to the right boundary vertices of the cylinder. We show that, after invoking the Gauss law on these vertices at the boundary, only one survives and enters in the definition of the vertical links. Before that, we can define the vertical links also according the same rule used for bulk vertical links

$$\sigma_{(L+1,y),\hat{2}}^x = \mu_{(L+1,y)}^z \mu_{(L,y)}^z. \quad (\text{A14})$$

Starting with $v = (L+1, 1)$ we use the Gauss law

$$G_{(L+1,1)} = \sigma_{(L+1,1),\hat{1}}^x \sigma_{(L+1,1),\hat{2}}^x \sigma_{(L+1,1),-\hat{2}}^x = q_{(L+1,1)} \quad (\text{A15})$$

where $\sigma_{(L+1,1),-\hat{2}}^x = \sigma_{(L+1,N),\hat{2}}^x$ which takes into account the extra symmetry of the system due to the use of *periodic boundary conditions*. Thus, substituting with (A9) and (A14) we obtain

$$\begin{aligned} \sigma_{(L+1,1),\hat{2}}^x &= q_{(L+1,1)} \eta(L, 1) \mu_{(L,1)}^z \mu_{(L,N)}^z \mu_{(L,N)}^z \mu_{(L+1,N)}^z \\ &= \left(\prod_{j=1}^{L+1} q_{(j,1)} \right) \mu_{(L,1)}^z \mu_{(L+1,N)}^z \end{aligned} \quad (\text{A16})$$

We can move to the vertex $v = (L+1, 2)$

$$G_{(L+1,2)} = \sigma_{(L+1,2),\hat{1}}^x \sigma_{(L+1,2),\hat{2}}^x \sigma_{(L+1,1),\hat{2}}^x = q_{(L+1,2)}. \quad (\text{A17})$$

and see a pattern coming out using the previous vertical link in (A16) to get

$$\begin{aligned} \sigma_{(L+1,2),\hat{2}}^x &= q_{(L+1,2)} \sigma_{(L+1,2),\hat{1}}^x \sigma_{(L+1,1),\hat{2}}^x \\ &= \left(\prod_{k=1}^2 \prod_{j=1}^{L+1} q_{(j,k)} \right) \mu_{(L,2)}^z \mu_{(L+1,N)}^z. \end{aligned} \quad (\text{A18})$$

We can now generalize for all the rightmost vertical links

$$\sigma_{(L+1,y),\hat{2}}^x = \left(\prod_{k=1}^y \prod_{j=1}^{L+1} q_{(j,k)} \right) \mu_{(L,y)}^z \mu_{(L+1,N)}^z. \quad (\text{A19})$$

Notice that for $y = N$, and using (A14), we obtain the conservation of charges, that is the product of all the

static charges q_v is one

$$\prod_{k=1}^N \prod_{j=1}^{L+1} q_{(j,k)} = 1. \quad (\text{A20})$$

Another interesting fact is that the last vertical links need non-local interactions $\mu_{(L,y)}^z \mu_{(L+1,N)}^z$.

Putting together all these results, we obtain the final dual Hamiltonian

$$\begin{aligned} H_{\text{dual}} &= -\frac{1}{g} \sum_{v \in \Lambda} \mu_v^x - g \sum_{\nu=1,2} \sum_{v \in \text{int } \Lambda} \omega(v, \hat{\nu}) \mu_v^z \mu_{v-\hat{\nu}}^z \\ &\quad - g \sum_y \left(\mu_{(1,y)}^z + \left(\prod_{k=1}^y \omega_{(L+1,k),\hat{2}} \right) \mu_{(L,y)}^z \mu_{(L+1,N)}^z \right) \end{aligned} \quad (\text{A21})$$

We remark on the following important fact: the information about the static charges is not encoded in the degrees of freedom of the dual model per se, but only in the couplings of the interaction term $\mu^z \mu^z$. The extra phase factor that appears in this term carries the content of the Gauss law at each site, as shown by (A13).

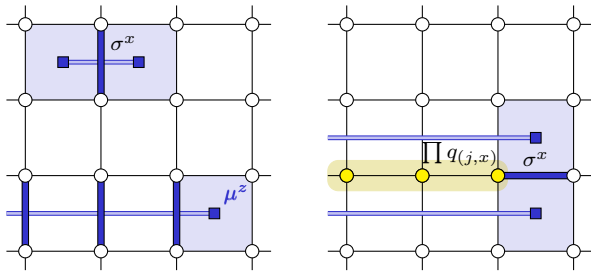
The last point left up for discussion is the role of the additional qubit $\mu_{(L+1,N)}^z$ and the topological sectors of the gauge model. The key insight is the fact that the plaquettes states do not actually fully specify a physical state, due to the possibility of non-contractible electric loops on the cylinder. An electric string is a path along which $\sigma^x = -1$, and a loop is just a closed string, without need of charges. A non-contractible electric loop is an electric loop that goes around the cylinder and cannot be removed via plaquette operators. It is possible to define an indicator for these non-contractible loops. Consider the string operator at vertical position y :

$$S_y = \prod_{j=1}^{L+1} \sigma_{(j,y),\hat{2}}^x = \left[\prod_{k=1}^y \prod_{j=1}^{L+1} q_{(j,k)} \right] \mu_{(L+1,N)}^z, \quad (\text{A22})$$

The product in brackets is a sign factor that depends on the number of charges below the string in y . The values of S_y are either $+1$ or -1 in the vacuum sector, are independent of y , and are invariant under the application of gauge-invariant operators. Hence, the different eigenspaces of S_y are the different topological sectors of the gauge mode. Using (A3), (A4), and (A19) we obtain

$$S_y = \mu_{(L+1,N)}^z = \mu_A^z, \quad (\text{A23})$$

meaning that the value of the ancilla depends on the topological sector.



Supplementary Figure 1. **Graphical sketch of the dual transformation of electric field operators.** In a dual formulation of the $\mathbb{Z}_2$ LGT to the Ising model, electric field terms map to nearest neighbor interaction in the transversal direction. In particular, electric fields on horizontal links (*right*) depend on products of charges along a horizontal path (highlighted in yellow).

Supplementary Note B: Finite-size scaling for the deconfining transition

The gauge Hamiltonian¹ has two phases in the vacuum, a confining one for strong coupling $g \gg 1$ and a deconfining one for weak coupling $g \ll 1$, with a second-order phase transition between them (see Figure 1 of the main text). In the confinement phase, the electric term dominates, and the excitations induced by the plaquettes are suppressed. Conversely, in the deconfinement phase, the magnetic term dominates, which, combined with Gauss' law, realizes a ground state that is a superposition of all the possible gauge-invariant configurations.

The vacuum phase diagram of the gauge model can be mapped onto the phase diagram of the quantum Ising model. The confining and deconfining phases, therefore, correspond to the ferromagnetic and paramagnetic phases, respectively. The duality also extends to the phase transition point. This means we can detect the critical point g_c of the gauge model by finding its dual in the Ising model. The order parameter of the dual Ising model is the magnetization

$$M = \frac{1}{N} \sum_x \langle \mu_x^z \rangle, \quad (\text{B1})$$

which has a second-order phase transition at the critical point. Therefore, we expect the magnetic susceptibility $\chi_M = dM/dg$ to diverge with a critical exponent γ :

$$\chi_M \sim |g - g_c|^{-\gamma}. \quad (\text{B2})$$

This, however, is true only in the thermodynamic limit $L \rightarrow \infty$. At finite size, we see finite peaks of height $\chi_c^{(L)}$ at positions $g_c^{(L)}$. The latter can be considered as pseudo-critical points and shifted to the true value g_c .

We avoid boundary effects on the magnetization by considering only sites in the bulk of the system.

We can apply the following reasoning to extract the true critical point. The correlation length ξ is also expected to diverge at the critical point with critical exponent ν ,

$$\xi \sim |g - g_c|^{-\nu}. \quad (\text{B3})$$

But it is hindered by the finite size, meaning that $\xi \sim L$. For this reason we replace ξ with L inside (B3) and invert the equation to obtain $|g - g_c| \sim L^{-1/\nu}$. Therefore, we can assume that the pseudo-critical points $g_c^{(L)}$ follow the relation

$$g_c^{(L)} = aL^{-1/\nu} + c, \quad (\text{B4})$$

which we can then fit against our numerical data and extrapolate g_c as

$$g_c = g_c^{(\infty)} = \lim_{L \rightarrow \infty} g_c^{(L)}. \quad (\text{B5})$$

The result of this procedure is shown in the bottom section of Suppl. Fig. 2, where we have obtained $g_c^M = 0.57(1)$. This value is compatible with the known result $g_c^{\text{th}} = 0.5731$ available in literature [1].

It is important to point out that in the gauge model, M corresponds to the expectation values of 't Hooft strings, i.e., the product of strings of σ^x along paths in the dual lattice, which makes it a *non-local* order parameter from the gauge model point of view.

Another method of finding the critical point is by measuring the entanglement entropy (EE), which is defined as

$$S = -\text{Tr} \rho_A \log \rho_A, \quad (\text{B6})$$

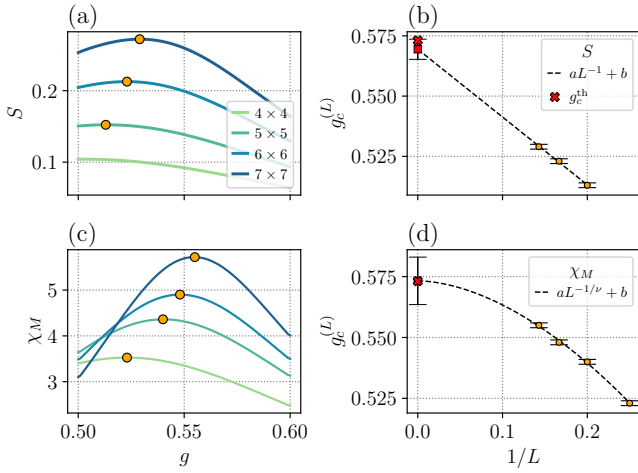
where ρ_A is the reduced density matrix $\rho_A = \text{Tr}_B \rho$, with $A \cup B$ being the bipartition of the whole system. In our case, we simply cut the system in the middle for the bipartition. Both the ferromagnetic and paramagnetic phases are gapped, with a finite correlation length $\xi < L$, meaning that the entanglement saturates quickly with the system size. On the other hand, at the critical point, due to $\xi \rightarrow \infty$, we expect the entanglement to grow, especially with increasing system size, and to saturate to a much higher value when compared to the gapped phases. Therefore, we expect a maximum in S at the critical point.

Similarly to before, with the positions $g_c^{(L)}$ of the EE maxima and finite-size scaling, we can extract the true value of g_c . As before, we can extract g_c by extrapolating for $L \rightarrow \infty$ with an appropriate scaling law, and the data suggest a simple relation

$$g_c^{(L)} = g_c^\infty + \frac{a}{L}. \quad (\text{B7})$$

We can see the outcome in the top panel of Suppl. Fig. 2, where we have obtained a value $g_c^{\text{EE}} = 0.570(4)$, again compatible with available literature.

¹ See Eq. (1) of the main text for reference.



Supplementary Figure 2. **Finite-size scaling of entanglement entropy S and magnetic susceptibility χ_M .** Left: S and χ_M ((a) and (c) respectively) around the critical point for $L = 4, \dots, 7$. The EE is computed for a half-lattice bipartition, while the magnetization is evaluated only in the bulk (square region with side $\lceil L/2 \rceil$). In both cases we used 101 points for $g \in [0.5, 0.6]$. Right: the extrapolation for $L \rightarrow \infty$ of the pseudo-critical points $g_c^{(L)}$, both for S (b) and χ_M (d). From the EE we find $g_c^{\text{EE}} = 0.570(4)$, while from the susceptibility $g_c^{\text{M}} = 0.57(1)$ (with $\nu = 0.57 \pm 0.17$). Both values of g_c are compatible with the known value $g_c^{\text{th}} = 0.5731$ that comes from the literature [1].

We remark that we are not looking for an accurate estimation of the deconfining critical point, hence the rudimentary procedure. We only need a value precise enough to distinguish the second-order phase transition of the gauge model from the roughening transition of the flux string. Furthermore, the use of square lattices makes the extrapolation to the thermodynamic limit much more sensible than doing so with rectangular lattices.

Supplementary Note C: Higher-order corrections to the potential

Besides the Lüscher term², other higher-order corrections to the potential can be obtained through the proper modeling of the flux string. But the latter has not been an easy task, especially if one intends to quantize the model. For example, the simplest proposal is the Nambu-Goto action [2, 3] and it is Lorentz-invariant only at the critical dimension $D = 26$. Another example is the Polyakov action [4], whose quantization leads to an extra oscillatory mode. The most accepted proposal for a quantum model of a quantized string comes from the Polchinski-Strominger action [5], which is an effective model valid only for long strings.

The spectrum for an open string with Dirichlet boundary conditions of the Polchinski-Strominger model is [6]

$$E_n = \sigma r \sqrt{1 - \frac{\pi}{12\sigma r^2}(D-2) + \frac{2\pi n}{\sigma r^2}}. \quad (\text{C1})$$

By expanding the square root for $n = 0$ (the ground state), one can find the higher-order corrections to the potential. At order r^{-3} one obtains

$$E_0(r) = \sigma r - \frac{\pi(D-2)}{24r} - \frac{\pi^2(D-2)^2}{1152\sigma r^3} + O(r^{-5}). \quad (\text{C2})$$

Notice that the Lüscher term is recovered plus an extra term δ/r^3 , where

$$\delta = -\frac{\pi^2(D-2)^2}{1152\sigma}. \quad (\text{C3})$$

Using the same setup used in the main text section about the confining potential, in particular the part about the confining potential, an attempt to fit the correction term (C3) has been made, but the results have not been promising, as can be seen in Suppl. Fig. 3. In this regard, we recognize the following issue. First, considering that (C3) is obtained from the effective action of a long continuous string, we suspect that the finite-size and lattice effects in the present setup are still very strong for such a high-order term. Second, similarly to the Lüscher term, the effective speed of sound v_s of the string must enter the Hamiltonian description and therefore also in (C3) in some, yet to be determined, way.

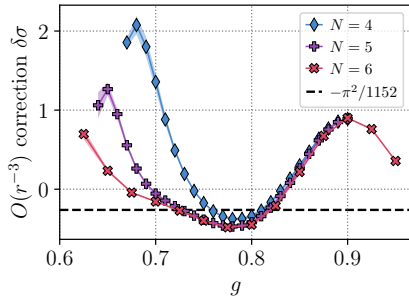
Supplementary Note D: Scaling of the kink mass and estimation of g_r

Following the definition of Eq. (21) of the main text, we have computed the kink mass m_k in the confined phase in a 30×5 lattice, for separations $R = 10, 11, \dots, 20$ and for different couplings. Based on the literature [7–10], we expect the roughening point to be in the region $0.8 \lesssim g \lesssim 1.0$. Hence, we have used 41 points in interval $[0.8, 1.0]$, while only 11 points in $[1.0, 2.0]$. The kink mass as a function of R is shown in Suppl. Fig. 4 only for some values of the coupling. From observing the data, we have assumed that at finite sizes, m_k scales as

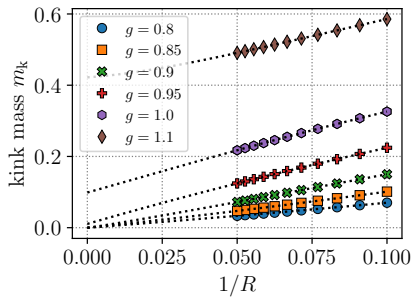
$$m_k(R) = m_k(\infty) + \frac{a_1}{R} + \frac{a_2}{R^2} + \dots \quad (\text{D1})$$

For this reason, it has been fitted against a high-order polynomial in $1/R$ (in this case, fifth-order). Then, an estimation of the limit value of $m_k(R \rightarrow \infty)$ is simply given by the constant term of the polynomial. The results of this fit are shown in Suppl. Fig. 4 with a dotted line for some values of g , while the overall shape of $m_k(R \rightarrow \infty)$ is depicted in Figure 6e of the main text. The errors on m_k from this extrapolation are, on average, in the order of 10^{-5} .

² See Eqs. (17–18) in the main text for reference.



Supplementary Figure 3. **Third order correction to the static potential.** The $O(r^{-3})$ correction term δ times σ of (C3) on a lattice with $L = 30$. Data of the potential for $r = 7, 8, \dots, 21$ have been used for the fit.



Supplementary Figure 4. **Scaling of the kink mass as a function of $1/R$ for different values of the coupling.** The dotted lines represent the fit with a fifth-degree polynomial in $1/R$.

The gap Δ near a BKT transition g_{BKT} is known to exponentially close as [11]

$$\Delta \sim A \exp \left(-\frac{b}{\sqrt{g - g_{\text{BKT}}}} \right), \quad (\text{D2})$$

when coming from the massive phase (A and b are constant to be determined), whereas Δ vanishes in the critical phase in thermodynamic limit. If we suppose that m_k indeed quantifies the gap *only in the massive phase*, as argued in main text section about the restoration of rotational symmetry, then from comparing $m_k(R \rightarrow \infty)$ to (D2) we should be able to get an estimation of the roughening point $g_r = g_{\text{BKT}}$. In order to do so, we need to be close enough to the roughening point, but without falling into the critical phase. For this exact reason that we select only the points in the interval $g \in [0.9, 1.0]$. The result of the fit is shown in Figure 6f of the main text, from we have found we find $g_r = 0.907(1)$, $A = 11(2)$, and $b = 1.44(6)$.

We warn that the precision about the roughening point g_r should be taken with a grain of salt. Due to the nebulous relation between the kink mass and the real excitation gap of the string in the vicinity of the transition point, it is still unclear what the proper finite-size scaling procedure should be applied to m_k . For example, we

have been unable to apply the techniques developed in [12, 13], since m_k does not reflect the gap in the critical phase. Second, it should be noted that the extrapolation of $m_k(R \rightarrow \infty)$ yields in the interval $[0.8, 0.925]$ slightly negative values in the order of -10^{-3} to -10^{-4} with an error in the order of 10^{-5} . The magnitude of this error is sensitive to the order of the polynomial used in (D1). We suspect that the issue lies with logarithmic corrections to the gap at the BKT point [14].

Supplementary Note E: Numerical methods analysis

In MPS simulations, a typical, problem-agnostic source of error is the approximation of the bond dimension χ . Nevertheless, our specific problem under study requires additional considerations. The finiteness of the system considerably influences the outcome of the experiments, even if performed in ideal settings. For this reason, we need to take into account finite size effects (as already explained in SN B). Since we embed our system in a 2D lattice, we have both transversal and longitudinal finite-size effects. The scaling in those two dimensions allows us to retrieve what would be the correct thermodynamic limit. Since we use MPS (a notoriously 1D technique) for a 2D lattice, we present different approaches to embed the lattice *dof* within the MPS sites. This study is motivated by the observation that different mappings can change the computational cost [15]. In particular, we compare our strip scheme (already shown in Figure 3 of the main text) with a more common one in literature [16], the snake, winding across the cylinder. Eventually, when studying the dynamics of the model, we have another source of numerical error due to the approximation of the evolution operator, that is, the Trotter error. In this section, we show all the techniques adopted to obtain the static and dynamic results with a consistent degree of reliability.

1. MPS mapping

The 2D system under study, when mapped to the Ising model, loses its long range character which was present on the direct lattice. Then, to represent the dual Hamiltonian with our MPS we need to further map it down to a 1D chain. It has been shown, [15] that preserving the locality of the 2D interactions into the TN structure is crucial to have an efficient numerical approach. A simple snake curve across the dual lattice results in a long range interaction model onto the 1D chain. Another approach, if our system is $L \times N$, is to group N sites forming an enlarged tensor and executing the DMRG on the final chain of L super-sites. The latter, is the method chosen in our research as also pointed out in the main text. Unfortunately, both methods have an exponential scaling in increasing the vertical size N , but the snake is more suited for studying quasi-2D systems. From the theory,

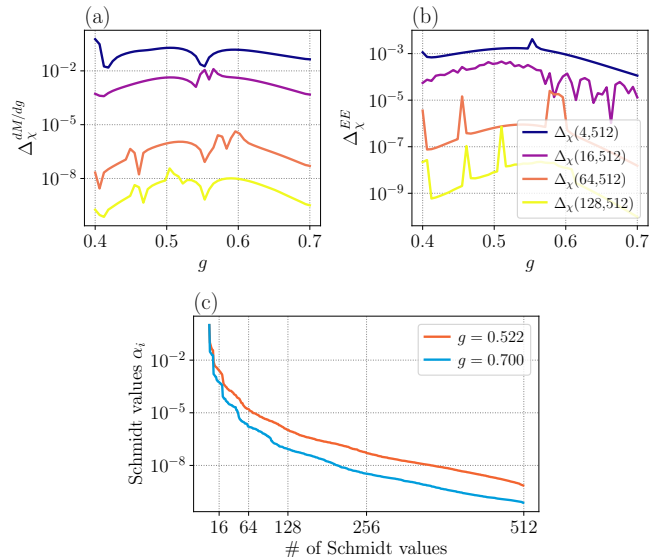
we are interested in a ladder system and this justifies blocking vertical slices of our cylinder. Our approach maintains the interactions to nearest neighbors and allows for an easier definition and interpretation of the string observables. In addition, when looking at the dynamics, the long range interactions will need many more layers than the one we use for trotterization. We heuristically find that a converging bond dimension for the strip is $\tilde{\chi} = 2^{N+2}$. A reason why this bond dimension leads us to convergence can be related exactly to the local nature of this mapping. This can be enough to study the rough physics of the model due to the fact that the bulk gauge theory is still massive and no abrupt change is shown in the entropy for finite strings. By performing a singular value decomposition (SVD) of the super-sites with $d = 2^N$ we show that we can recover the snake curve but with a higher (inhomogeneous) bond dimension, increasing up to $\chi = \tilde{\chi} 2^{N/2}$. Then, the cost increases and it can be comparable to that of the strip geometry revealing that none of the two methods really wins over the other in terms of computational cost.

2. Vacuum sector analysis

For the results of the vacuum sector presented in the previous SN B, we use square *obc* lattices of side $N = L = 4, 5, 6, 7$ and maximum bond dimension of $\chi = 64$. As the physical dimension of our MPS is, as shown in Figure 3 of the main text, 2^N , the larger the transversal size, the more entanglement will build up, and the larger should be the bond dimension to well represent the true ground state. Thus, we take the largest lattice of 7×7 to show its convergence in χ , implying this will be valid for smaller lattices. The convergence of the bond dimension depends on the observable we are analyzing, and can have different rates. Here, we are interested in the convergence for the two quantities dM/dg and the entanglement entropy S . To do so, we look at the difference between a maximum bond dimension and the others $\Delta_{\chi}^{\hat{O}}(\chi_{\max}, \chi)$ for the observable $\hat{O}$ we are interested in. The results in Suppl. Fig. 5 show that this difference goes to $10^{-5} - 10^{-7}$ already for bond dimension $\chi = 64$ that is the one used in the previous SN B to find the critical point g_c with great precision.

3. Two-particle sector analysis

The numerics of the two-particle sector makes use of cylindrical lattices, which allows us to reduce finite size effects in the transversal direction. The effective string symmetry is $U(1)$ in the thermodynamic limit, but for us the transversal fluctuations are truncated and effectively represented by a $\mathbb{Z}_N$ gauge group. Thus, as already seen for the entropy (inset of Figure 4c of the main text) and the string width (Figure 4a), the transversal size N highly influences the observed physics as $g \rightarrow g_c$, the deconfine-

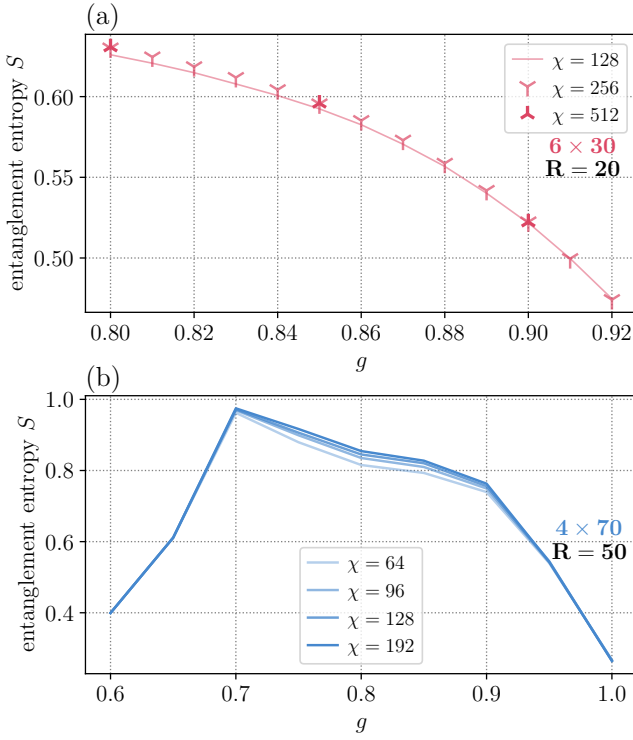


Supplementary Figure 5. **Ground state convergence in bond dimension.** Convergence in bond dimension of 't Hooft and EE for 6×6 lattice. **(a)** Difference between the maximum bond dimension 512 and smaller ones for the Magnetic Susceptibility. **(b)** Difference between the maximum bond dimension 512 and smaller ones for Entanglement Entropy (b). At $\chi = 64$, we have in both quantities a significant convergence of $10^{-5} - 10^{-7}$. We explain this difference in the order of magnitude because of the phase transition that we are crossing. **(c)** Schmidt values decay at two different electric couplings g . At the pseudo-critical point $g_c^{(L=6)}$, the system is “harder” to approximate than for other g and the tail of Schmidt values decays slower than, e.g., $g = 0.7$.

ment point. This is not true for all the coupling values g , as for some quantities the results are independent of N . Even though the results converge in N , most of the quantities we are interested in are valid for the limit of $R \rightarrow \infty$, and thus, need to converge in our longitudinal size L . Hence, fixing N , we show these two aspects in the following: convergence in bond dimension of the EE and w^2 for fixed charge separations R , and convergence to the asymptotic limit of the Lüscher term, central charge, and on-axis vs. off-axis comparisons varying R .

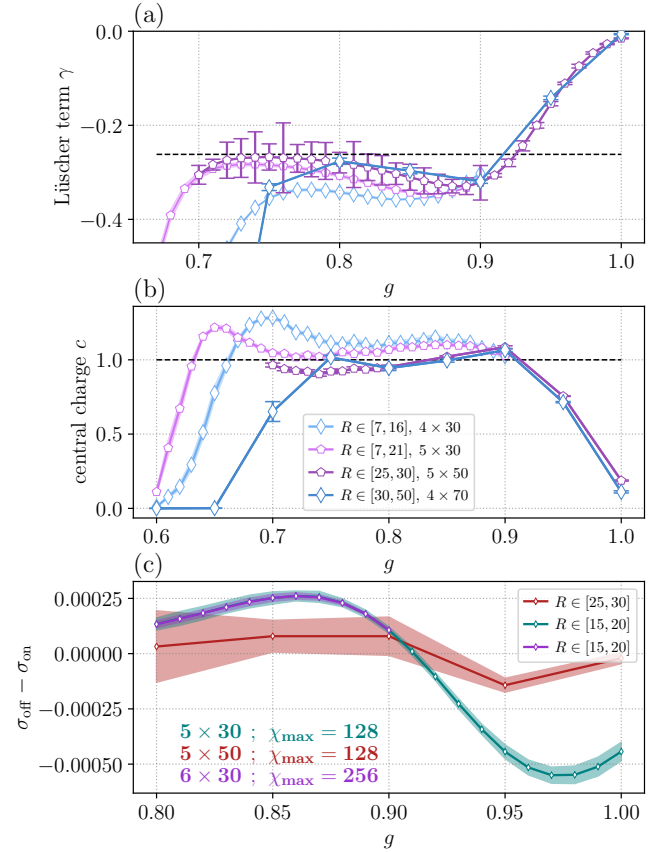
Firstly, for a lattice 6×30 we find a maximum relative error between the string widths $w^2(g = 0.625, R = 19, \chi = 256)$ and $w^2(g = 0.625, R = 19, \chi = 128)$ of 4×10^{-4} . Nonetheless, the entropy converges more slowly, and higher bond dimensions are needed (top panel of Suppl. Fig. 6). We show in the bottom panel of Suppl. Fig. 6 the EE for a cylinder 4×70 with a string of $R = 50$, and $\chi = 64, 96, 128, 192$. Here we can appreciate the computational hardness of the roughening region, that is, where the bond dimension struggles the most to converge.

Then, we address the convergence to the Lüscher term γ and central charge c that should be $-\pi/12$, and 1, respectively, in the crossover region. This analysis allows us to see that the extent of the roughening region depends



Supplementary Figure 6. **Convergence in bond dimension of the entanglement entropy.** We use lattices of 6×30 and 4×70 . (a) The lattice with transversal size $N = 6$ shows a convergence in bond dimension around 0.90 for $\chi = 128$. For $\chi = 256$ we see convergence with $\chi = 512$ for all the couplings $g > 0.8$. This is valid for $R = 20$. (b) The same procedure is applied for $N = 4$, but we take a larger longitudinal size $L = 70$ and maximum string of $R = 50$. Here we can appreciate the complexity of the roughening region where the χ required to correctly represent the ground state of a large string grows.

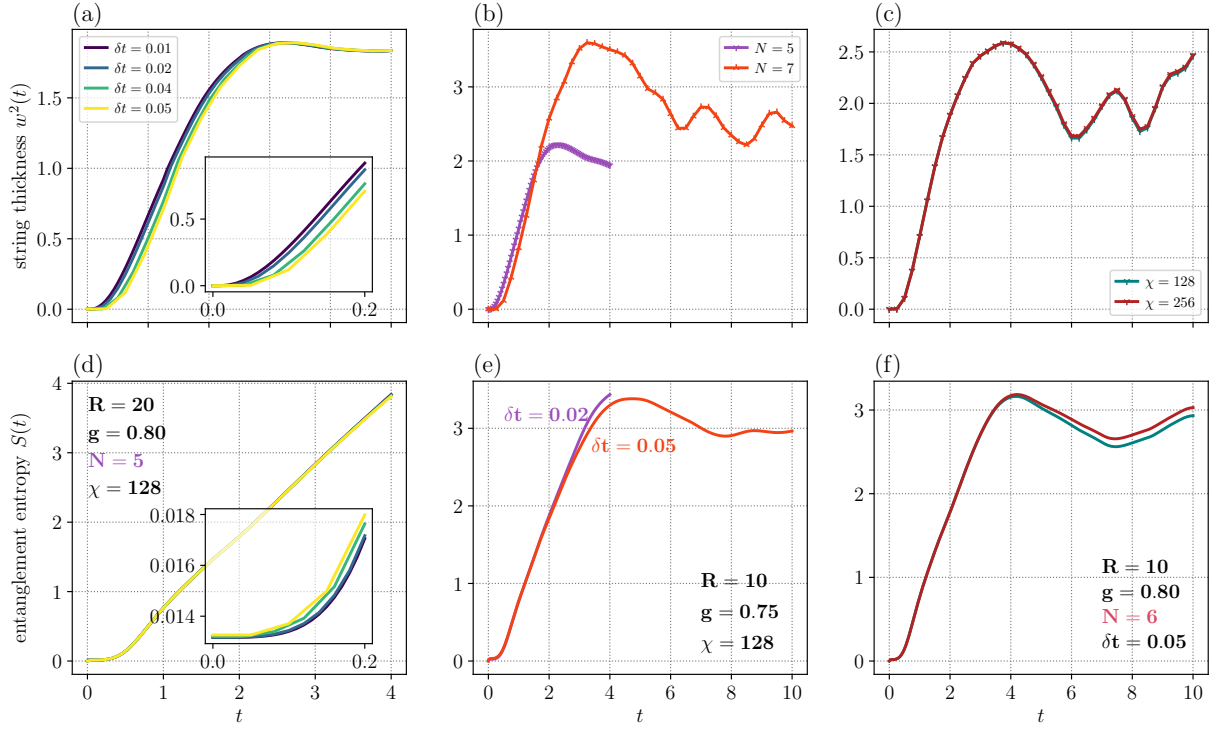
on the transversal size N as depicted in Suppl. Fig. 7. The larger the N , the more the plateaus in the Lüscher term and the central charge sweep couplings up to the thermodynamical limit of the deconfining point g_c , the actual critical point of our model. On the other hand, by increasing the length of the cylinder and the separations R considered in the fits to extract γ and c , we observe a flattening of the plateaus to the expected theoretical values. As a last remark, we also report the difference in the on-axis and off-axis string tensions $\sigma_{\text{off}} - \sigma_{\text{on}}$ at the onset of the critical region for the string. The oscillations present also in the right plot of Figure 5 of the main text are not due to the transversal size N (here it is clear that for $N = 5$ and $N = 6$ the result is unchanged), but they are caused by the short R considered for the fit of the on-axis and off-axis potentials.



Supplementary Figure 7. **Lüscher term γ , central charge c , and string tension difference for increasing transversal N and longitudinal L size.** (a) The Lüscher term γ is affected by both N and L . By increasing N , we increase the extent of the roughening region, by increasing L , and consequently R , the value of $\gamma(g)$ gets closer to the universal value (black dashed line). (b) The same results seem to apply also for the central charge c , which is extracted fitting for different R following Eq. (15) of the main text. The plateau gets closer to the bosonic string prediction of $c = 1$ (black dashed line) as we use larger strings with many transversal degrees of freedom available for fluctuations. (c) The difference in on-axis and off-axis potentials of the string tension σ . We change N and L and see that the onset of the roughening region depends more on the length R of the strings than on the transversal size N .

4. Dynamical error analysis

In this section, we see the main limitations of our TEBD algorithm. The Trotterization procedure can greatly affect the expectation values of our observables. From Suppl. Fig. 8a–d, we see that the string width varies by changing the δt , with a great influence at short times as shown in the inset, but the entanglement entropy converges faster. The transversal size also plays a big role in the saturation of the quantum fluctuations of the string. Letting N grow to 7, we see that the spreading of the string does not seem to stop for a value in the roughening



Supplementary Figure 8. **Error dynamics in EE and string width w^2 due to finite trotter step δt , transversal size N , and bond dimension χ .** On the upper part, we show the string thickness behavior for trotter steps $\delta t \in [0.01, 0.05]$ (a), transversal size N from 5 to 7 (b), and bond dimension χ from 128 to 256 (c). While for (a) and (b) we see that w^2 is affected, it seems to converge nicely with the bond dimension. On the lower part, we present the entanglement entropy in (d), (e), and (f) with the same parameters as above. Here, the entropy shows quite different behaviors. It is converging faster for changing δt and N , but does not converge in bond dimension for $t \gtrsim 4$.

region. This suggests that more studies on the transversal size are needed to understand when the string is saturating because of the coupling or for the finite transversal size. This is clear in Suppl. Fig. 8b–e where w^2 seems to saturate around $t = 2$ for $N = 5$, but the result for larger N clearly shows that this was a finite size effect. Contrarily, the entropy seems to follow a bit better the trend of larger lattices even though they are not converging. Finally, in Suppl. Fig. 8c–f, the result of changing the bond dimension, fixing the $\delta t = 0.05$ and the lattice to 30×6 shows no particular difference in the trends for a string of length $R = 10$.

SUPPLEMENTARY REFERENCES

- [1] H. W. J. Blöte and Y. Deng, Cluster monte carlo simulation of the transverse ising model, *Physical Review E* **66**, 10.1103/physreve.66.066110 (2002).
- [2] Y. Nambu, Qcd and the string model, *Physics Letters B* **80**, 372–376 (1979).
- [3] P. Goddard, J. Goldstone, C. Rebbi, and C. Thorn, Quantum dynamics of a massless relativistic string, *Nuclear Physics B* **56**, 109–135 (1973).
- [4] A. Polyakov, Fine structure of strings, *Nuclear Physics B* **268**, 406–412 (1986).
- [5] J. Polchinski and A. Strominger, Effective string theory, *Physical Review Letters* **67**, 1681–1684 (1991).
- [6] J. Kuti, Lattice qcd and string theory, *International Journal of Modern Physics A* **21**, 699–706 (2006).
- [7] A. Hasenfratz, E. Hasenfratz, and P. Hasenfratz, Generalized roughening transition and its effect on the string tension, *Nuclear Physics B* **180**, 353–367 (1981).
- [8] J. B. Kogut, D. K. Sinclair, R. B. Pearson, J. L. Richardson, and J. Shigemitsu, Fluctuating string of lattice gauge theory: The heavy-quark potential, the restoration of rotational symmetry, and roughening, *Physical Review D* **23**, 2945–2961 (1981).
- [9] W. Krinitsin, N. Tausendpfund, M. Rizzi, M. Heyl, and M. Schmitt, *Roughening dynamics of interfaces in two-dimensional quantum matter* (2024), arXiv:2412.10145 [quant-ph].
- [10] W.-T. Xu, M. Knap, and F. Pollmann, *Tensor-network study of the roughening transition in (2 + 1)d lattice gauge theories* (2025), arXiv:2503.19027 [cond-mat.str-el].
- [11] J. M. Kosterlitz, The critical properties of the two-dimensional xy model, *Journal of Physics C: Solid State Physics* **7**, 1046–1060 (1974).
- [12] T. Mishra, J. Carrasquilla, and M. Rigol, Phase diagram of the half-filled one-dimensional $t - V - V'$ model, *Physical Review B* **84**, 10.1103/physrevb.84.115135 (2011).
- [13] M. Dalmonte, J. Carrasquilla, L. Taddia, E. Ercolessi, and M. Rigol, Gap scaling at berezinskii-kosterlitz-

- thouless quantum critical points in one-dimensional hubbard and heisenberg models, *Physical Review B* **91**, [10.1103/physrevb.91.165136](#) (2015).
- [14] F. Woynarovich and H. P. Ecker, Finite-size corrections for the low lying states of a half-filled hubbard chain, *Journal of Physics A: Mathematical and General* **20**, L443–L449 (1987).
 - [15] G. Cataldi, A. Abedi, G. Magnifico, S. Notarnicola, N. D. Pozza, V. Giovannetti, and S. Montangero, Hilbert curve vs Hilbert space: exploiting fractal 2D covering to increase tensor network efficiency, *Quantum* **5**, 556 (2021).
 - [16] E. Stoudenmire and S. R. White, Studying two-dimensional systems with the density matrix renormalization group, *Annual Review of Condensed Matter Physics* **3**, 111 (2012).