

Supplementary Information for "Parameter Estimation for Model-Based Sensing of Magneto-Mechanical Resonators"

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Supplementary Note 1

In this section, the explicit expression (4) of the STFT for the signal $s_{j,n}(t) = \frac{1}{2}\alpha_{j,n}e^{i(\omega_{j,n}t+\psi_{j,n})}$ is derived. Inserting the signal into the STFT definition (3) yields

$$\begin{aligned} S_w^{j,n}(t, \tilde{\omega}) &= \int_{-\infty}^{\infty} s_{j,n}(\tau) w(\tau - t) e^{-i\tilde{\omega}\tau} d\tau \\ &= \frac{\alpha_{j,n}}{2} e^{i\psi_{j,n}} \int_{-\infty}^{\infty} e^{i\omega_{j,n}\tau} w(\tau - t) e^{-i\tilde{\omega}\tau} d\tau. \end{aligned}$$

Substituting $\tau' := \tau - t$ leads to

$$\begin{aligned} S_w^{j,n}(t, \tilde{\omega}) &= \frac{\alpha_{j,n}}{2} e^{i\psi_{j,n}} \int_{-\infty}^{\infty} e^{i\omega_{j,n}(\tau'+t)} w(\tau') e^{-i\tilde{\omega}(\tau'+t)} d\tau' \\ &= \frac{\alpha_{j,n}}{2} e^{i((\omega_{j,n}-\tilde{\omega})t+\psi_{j,n})} \int_{-\infty}^{\infty} e^{i\omega_{j,n}\tau'} w(\tau') e^{-i\tilde{\omega}\tau'} d\tau' \\ &= \frac{\alpha_{j,n}}{2} e^{i((\omega_{j,n}-\tilde{\omega})t+\psi_{j,n})} \mathcal{F}\{e^{i\omega_{j,n}\cdot} w(\cdot)\}(\tilde{\omega}). \end{aligned}$$

Here, we consider the non-unitary, angular frequency Fourier transform $\mathcal{F}\{a\}(\tilde{\omega}) := \int_{-\infty}^{\infty} a(t) e^{-i\tilde{\omega}t} dt$ with its inverse $\mathcal{F}^{-1}\{\mathcal{F}\{a\}\}(t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}\{a\} e^{i\tilde{\omega}t} d\tilde{\omega}$. Using the convolution theorem $\mathcal{F}\{ab\}(\tilde{\omega}) = \frac{1}{2\pi} \mathcal{F}\{a\} * \mathcal{F}\{b\}(\tilde{\omega})$ and with $\mathcal{F}\{e^{i\omega_{j,n}\cdot}\}(\tilde{\omega}) = 2\pi\delta(\tilde{\omega} - \omega_{j,n})$, where δ is the Dirac delta distribution, and $\mathcal{F}\{w\}(\tilde{\omega}) = W(\tilde{\omega})$, we obtain

$$\begin{aligned} S_w^{j,n}(t, \tilde{\omega}) &= \frac{\alpha_{j,n}}{2} e^{i((\omega_{j,n}-\tilde{\omega})t+\psi_{j,n})} \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\omega' - \omega_{j,n}) W(\tilde{\omega} - \omega') d\omega' \\ &= \frac{\alpha_{j,n}}{2} e^{i((\omega_{j,n}-\tilde{\omega})t+\psi_{j,n})} W(\tilde{\omega} - \omega_{j,n}), \end{aligned}$$

which completes the proof.

Supplementary Note 2

This section contains a brief derivation of the relationship in equation (18) between maximum angular velocity $\dot{\varphi}_{\max}$ and maximum deflection angle $\varphi_{\max}$. The MMRs considered in this work are 1D magneto-mechanical systems. Their rotational energy is given by

$$E_{\text{rot}} = \frac{1}{2} I \dot{\varphi}^2,$$

where I is the rotor's moment of inertia around the rotation axis and $\dot{\varphi}$ is its angular velocity. Assuming dipole-dipole magnetic interactions, the magnetic energy of the system, taking the equilibrium position as the reference state, is given by

$$E_{\text{mag}} = m_r B (1 - \cos \varphi),$$

where m_r is the magnetic moment of the rotor, B is the magnetic field of the stator at the rotors position, and φ is the deflection angle from the system's equilibrium position. For the states of maximum deflection and maximum angular velocity, the total energy of the system is given by

$$\text{I. } E_{\text{rot}}(\dot{\varphi}_{\text{max}}) + 0 = \text{const} \quad (\text{S1})$$

$$\text{II. } 0 + E_{\text{mag}}(\varphi_{\text{max}}) = \text{const.} \quad (\text{S2})$$

With (S1) and (S2) the two energies relate φ_{max} and $\dot{\varphi}_{\text{max}}$

$$\frac{1}{2} I \dot{\varphi}_{\text{max}}^2 = m_r B (1 - \cos \varphi_{\text{max}}).$$

The natural frequency is given by $\omega_{\text{nat}} = \sqrt{\frac{m_r B}{I}}$, as defined in the supplemental material of [1], which allows us to express the relation as

$$\dot{\varphi}_{\text{max}} = \omega_{\text{nat}} \sqrt{2 - 2 \cos(\varphi_{\text{max}})}$$

for the amplitude of the maximum angular velocity.

Supplementary Note 3

In this section, we derive the Fourier series (26) for the periodic deflection angle $\varphi^{\text{TU}}(t)$. According to [2, Eq. 31], an explicit solution to (23) can be expressed as

$$\varphi^{\text{TU}}(t) = 2 \arcsin \left(k \operatorname{sn} \left(K(k) - \frac{2K(k)}{\pi} (\omega^{\text{TU}}(\varphi_{\text{max}})t + \psi_{\text{start}}); k \right) \right),$$

where sn denotes the sinus amplitudinis, which is one of the fundamental elliptic functions [3] and $k := \sin(\varphi_{\text{max}}/2)$ denotes the elliptic modulus that we introduce to abbreviate notation. $K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 \Psi}} d\Psi$ denotes the complete elliptic integral of the first kind. In contrast to [2, Eq. 31] we consider a non-zero initial phase and therefore added ψ_{start} .

Substituting $x := \frac{2K(k)}{\pi} (\omega^{\text{TU}}(\varphi_{\text{max}})t + \psi_{\text{start}})$ and using the identity $\operatorname{sn}(x + K; k) = \operatorname{cd}(x; k)$ [4, Table 22.4.3] leads to

$$\varphi^{\text{TU}}(x) = 2 \arcsin(k \operatorname{cd}(-x; k)). \quad (\text{S3})$$

As the function cd is even, we can neglect the minus sign. We can then take the integral [4, §22.14.5]

$$kk' \int \operatorname{sd}(x'; k) dx' = \arcsin(-k \operatorname{cd}(x'; k)) + C \quad (\text{S4})$$

with $k' := \sqrt{1 - k^2}$. Using the property of $\arcsin$ being odd and applying the limits $K(k)$ and x , we can rewrite the indefinite integral (S4) as

$$-kk' \int_{K(k)}^x \operatorname{sd}(x'; k) dx' = \arcsin(k \operatorname{cd}(x; k)). \quad (\text{S5})$$

Inserting (S5) into (S3), we obtain

$$\varphi^{\text{TU}}(x) = -2kk' \int_{K(k)}^x \operatorname{sd}(x'; k) dx'. \quad (\text{S6})$$

The function $\operatorname{sd}(x'; k)$ can be written as a Fourier series using [4, §22.11.5]

$$\operatorname{sd}(x'; k) = \frac{2\pi}{K(k)kk'} \sum_{n=0}^{\infty} \frac{(-1)^n q(k)^{n+1/2}}{1 + q(k)^{2n+1}} \sin \left((2n+1) \frac{\pi}{2K(k)} x' \right)$$

with the nome

$$q(k) := \exp\left(-\pi \frac{K(k')}{K(k)}\right).$$

Inserting this into (S6) leads to

$$\varphi^{\text{TU}}(x) = -\frac{4\pi}{K(k)} \sum_{n=0}^{\infty} \frac{(-1)^n q(k)^{n+1/2}}{1 + q(k)^{2n+1}} \int_{K(k)}^x \sin\left((2n+1) \frac{\pi}{2K(k)} x'\right) dx'. \quad (\text{S7})$$

Solving the integral leads to

$$\begin{aligned} \int_{K(k)}^x \sin\left((2n+1) \frac{\pi}{2K(k)} x'\right) dx' &= \left[-\frac{2K(k)}{(2n+1)\pi} \cos\left((2n+1) \frac{\pi}{2K(k)} x'\right) \right]_{K(k)}^x \\ &= -\frac{2K(k)}{(2n+1)\pi} \left(\cos\left((2n+1) \frac{\pi}{2K(k)} x\right) - \underbrace{\cos\left((2n+1) \frac{\pi}{2}\right)}_{=0} \right). \end{aligned} \quad (\text{S8})$$

Inserting (S8) into (S7) and back-substituting x , we obtain

$$\varphi^{\text{TU}}(t) = 8 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \frac{q(k)^{n+1/2}}{1 + q(k)^{2n+1}} \cos\left((2n+1)(\omega^{\text{TU}}(\varphi_{\max})t + \psi_{\text{start}})\right). \quad (\text{S9})$$

Defining

$$c_n(sk) := \frac{(-1)^n}{2n+1} \frac{q(k)^{n+1/2}}{1 + q(k)^{2n+1}},$$

we can write (S9) as

$$\varphi^{\text{TU}}(t) = 8 \sum_{n=0}^{\infty} c_n(k) \cos\left((2n+1)(\omega^{\text{TU}}(\varphi_{\max})t + \psi_{\text{start}})\right),$$

which is the same as (26) and thus completes the derivation.

Supplementary Note 4

In this section, we show that the induced signal can be expressed as (27) with amplitudes (28) and (29) when considering the undamped model with $c_n = 0$ for $n > 0$. Starting from (13) and inserting the basis functions (11) and (12) yields

$$\mathbf{s}^{\text{TU}}(t) = m_{\text{r}} \boldsymbol{\sigma}_v \vartheta_v(t) + m_{\text{r}} \boldsymbol{\sigma}_u \vartheta_u(t) = m_{\text{r}} \boldsymbol{\sigma}_v \dot{\varphi}(t) \cos(\varphi(t)) - m_{\text{r}} \boldsymbol{\sigma}_u \dot{\varphi}(t) \sin(\varphi(t)). \quad (\text{S10})$$

We then want to eliminate the sine and cosine terms and to this end use (23) to replace the sine term. To substitute the cosine term, we differentiate (23) with respect to time, yielding $\dot{\varphi}(t) \cos \varphi(t) = -\frac{1}{\omega_{\text{nat}}^2} \ddot{\varphi}(t)$. Inserting both expressions into (S10) yields

$$\mathbf{s}^{\text{TU}}(t) = -\frac{m_{\text{r}} \boldsymbol{\sigma}_v}{\omega_{\text{nat}}^2} \ddot{\varphi}(t) + \frac{m_{\text{r}} \boldsymbol{\sigma}_u}{\omega_{\text{nat}}^2} \dot{\varphi}(t) \dot{\varphi}(t).$$

Now, we can insert the series expression (25) and exploit that $c_n = 0$ for $n > 0$ such that

$$\begin{aligned} \mathbf{s}^{\text{TU}}(t) &= -\frac{8m_{\text{r}} \boldsymbol{\sigma}_v}{\omega_{\text{nat}}^2} c_0(\omega^{\text{TU}})^3 \sin(\omega^{\text{TU}}t + \psi_{\text{start}}) \\ &\quad + \frac{64m_{\text{r}} \boldsymbol{\sigma}_u}{\omega_{\text{nat}}^2} c_0^2(\omega^{\text{TU}})^3 \sin(\omega^{\text{TU}}t + \psi_{\text{start}}) \cos(\omega^{\text{TU}}t + \psi_{\text{start}}). \end{aligned}$$

Note that $c_n(\varphi_{\max})$, $\omega^{\text{TU}}(\varphi_{\max})$ and $q(\varphi_{\max})$ depend on $\varphi_{\max}$, but for the sake of readability, we've omitted $\varphi_{\max}$ in the equations.

Then, we exploit trigonometric expression $2 \sin(x) \cos(x) = \sin(2x)$

$$\mathbf{s}^{\text{TU}}(t) = -\frac{8m_r\sigma_v}{\omega_{\text{nat}}^2}c_0(\omega^{\text{TU}})^3 \sin(\omega^{\text{TU}}t + \psi_{\text{start}}) + \frac{32m_r\sigma_u}{\omega_{\text{nat}}^2}c_0^2(\omega^{\text{TU}})^3 \sin(2\omega^{\text{TU}}t + 2\psi_{\text{start}}).$$

Inserting $c_0 = \frac{\sqrt{q}}{1+q}$ from (26) leads to

$$\mathbf{s}^{\text{TU}}(t) = -\frac{8m_r\sigma_v}{\omega_{\text{nat}}^2} \frac{\sqrt{q}}{1+q} (\omega^{\text{TU}})^3 \sin(\omega^{\text{TU}}t + \psi_{\text{start}}) + \frac{32m_r\sigma_u}{\omega_{\text{nat}}^2} \frac{q}{(1+q)^2} (\omega^{\text{TU}})^3 \sin(2\omega^{\text{TU}}t + 2\psi_{\text{start}}).$$

We then replace $\psi^{\text{TU}}(t) = \omega^{\text{TU}}(\varphi_{\text{max}})t + \psi_{\text{start}}$ and insert the expression for ω^{TU} from (24) and obtain

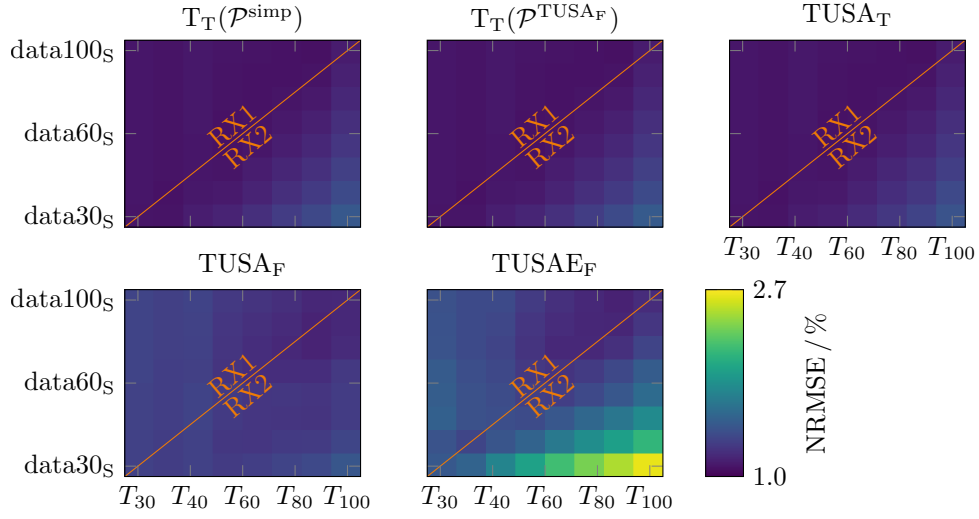
$$\mathbf{s}^{\text{TU}}(t) = -\underbrace{\frac{\pi^3\omega_{\text{nat}}m_r\sigma_v}{K(\sin(\varphi_{\text{max}}/2))^3} \frac{\sqrt{q}}{1+q}}_{\stackrel{(28)}{=} \boldsymbol{\alpha}_v(\varphi_{\text{max}})} \sin(\psi^{\text{TU}}(t)) + \underbrace{\frac{4\pi^3\omega_{\text{nat}}m_r\sigma_u}{K(\sin(\varphi_{\text{max}}/2))^3} \frac{q}{(1+q)^2}}_{\stackrel{(29)}{=} \boldsymbol{\alpha}_u(\varphi_{\text{max}})} \sin(2\psi^{\text{TU}}(t)),$$

which completes the proof.

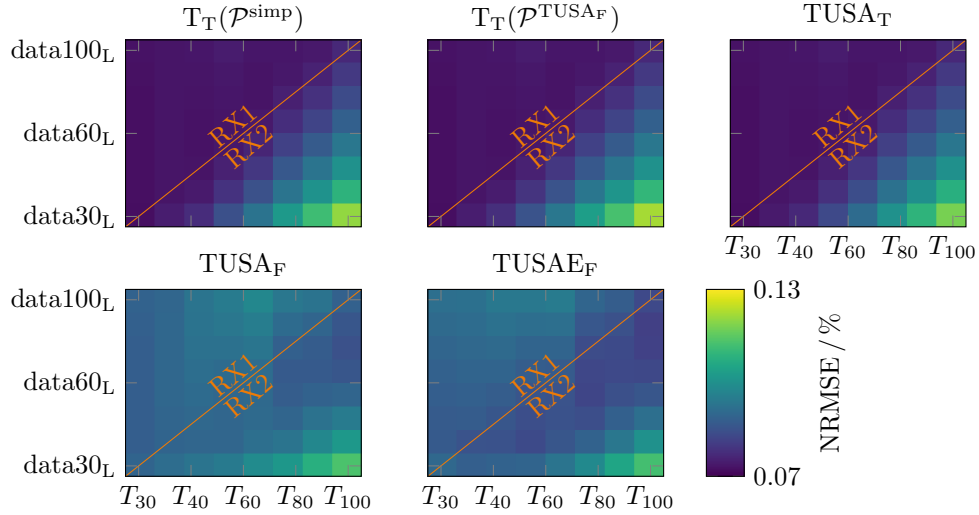
Supplementary References

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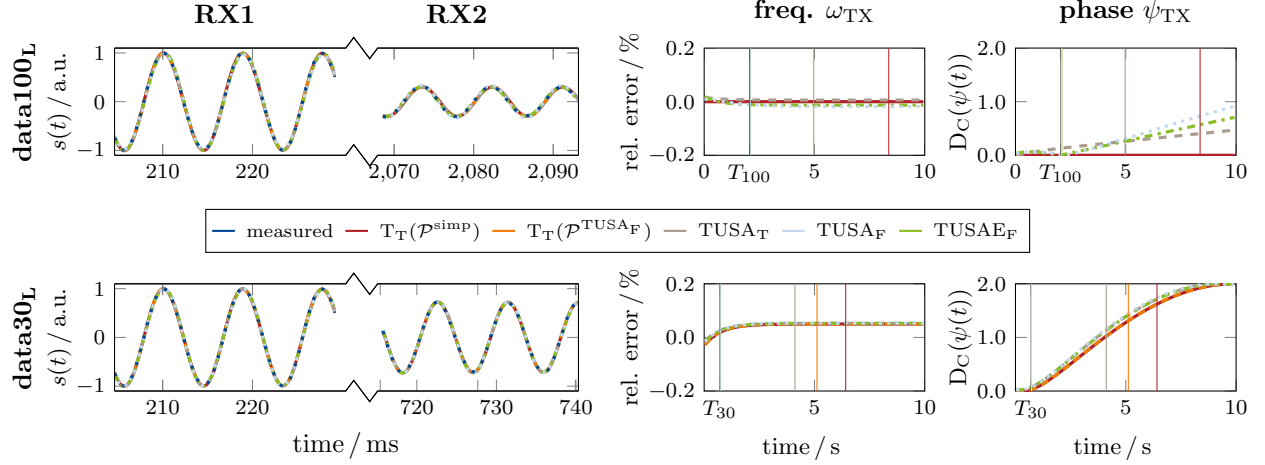
Supplementary Figures



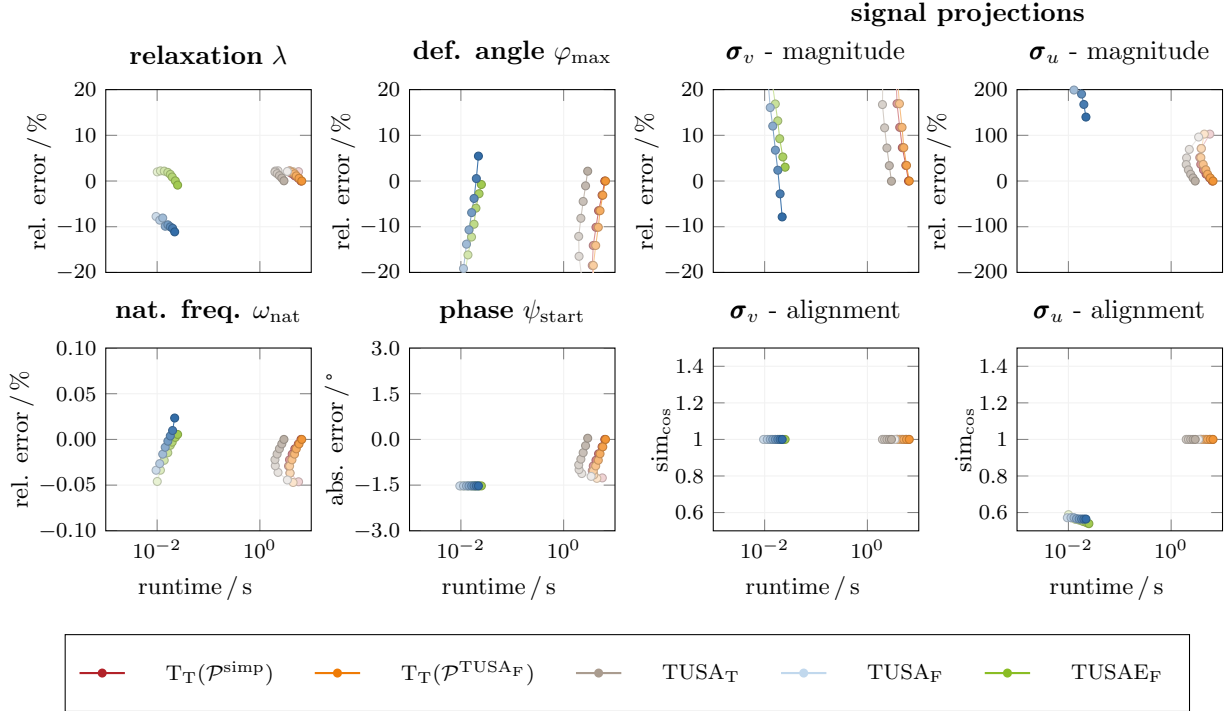
Supplementary Figure 1: Error to time signal; Experiment 1 - uncontrolled (MMR S). The normalized root mean square error for all methods over all frames are shown for different amounts of data and for different time ranges.



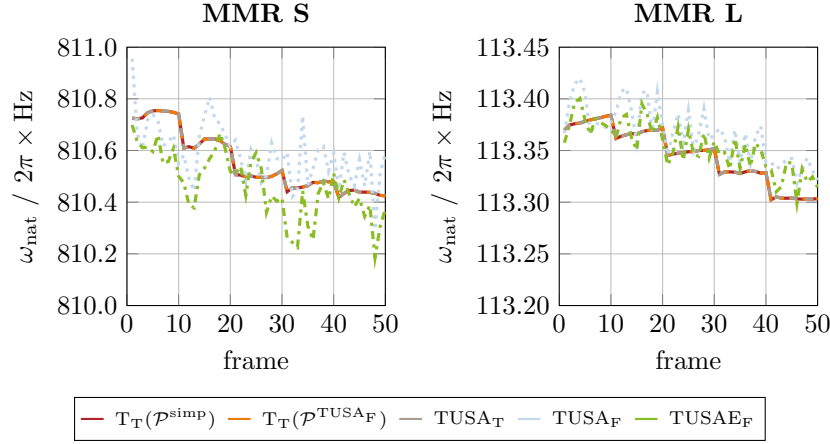
Supplementary Figure 2: Error to time signal; Experiment 1 - uncontrolled (MMR L). The normalized root mean square error for all methods over all frames are shown for different amounts of data and for different time ranges.



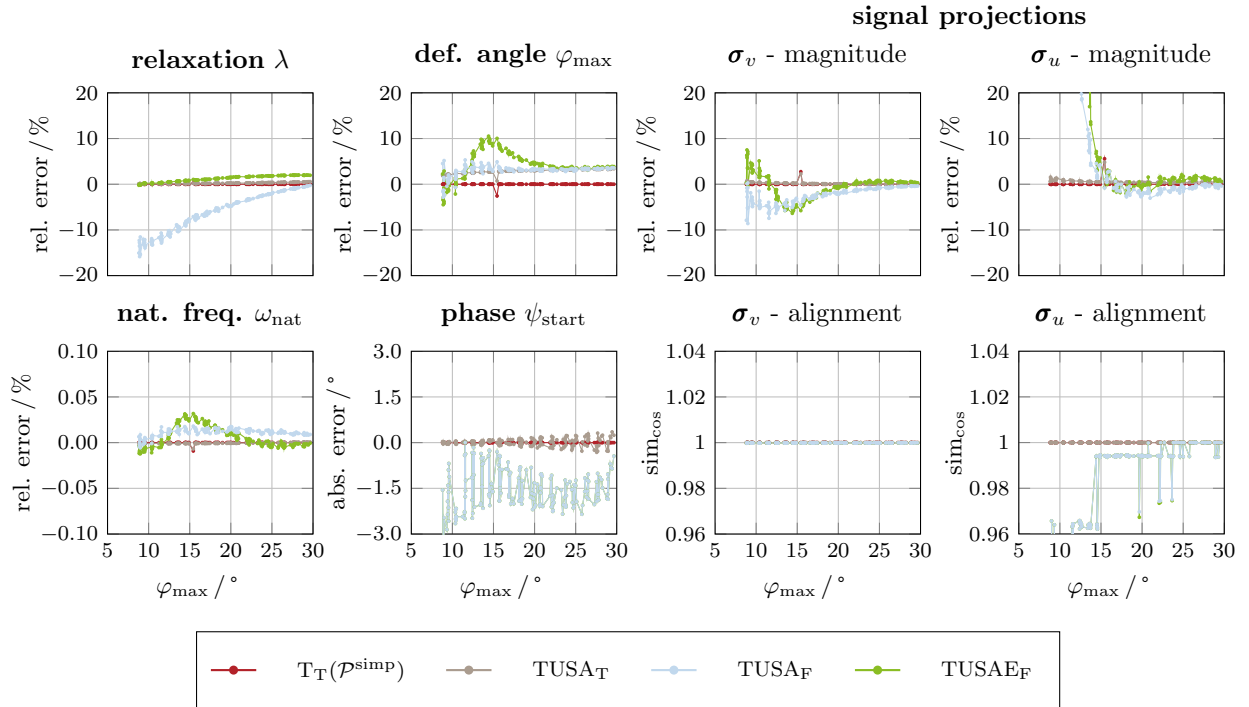
Supplementary Figure 3: Forecasts of time signal and control parameter; Experiment 1 - uncontrolled (MMR L). Left: Exemplary signal section at different time points of the measured signal in comparison to the estimated signals of all methods (excitation amplitude: 35 μ T). Signals for a time segment during RX1 as well as 100 ms after T_{30}/T_{100} (labeled RX2) are shown. Note that all methods reconstruct the signal very closely to the measured signal within RX1. Right: The deviation for all methods from the reference method for data30L and data100L for the forecast of frequency ω_{TX} and phase ψ_{TX} over time is displayed. Additionally, the cut-off time T_{30} , T_{100} as well as the mean runtime of the methods are shown in straight lines.



Supplementary Figure 4: Runtime and deviation for different datasets; Experiment 1 - uncontrolled (MMR L). The median deviation between the different methods and the reference method of all frames over the runtime for the truncated receive windows. For the parameters λ , $\varphi_{\max}$, ω_{nat} , σ_v and σ_u , the deviation is quantified using the relative error, for ψ_{start} using the absolute error, and for the alignment of σ_v and σ_u using the cosine similarity. The color fading in the line plots indicates the amount of data used, with the lightest shade representing data30L and the darkest data100L.



Supplementary Figure 5: Drift in estimated natural frequency; Experiment 1 - uncontrolled (MMR S/MMR L). The estimated natural frequency from the different methods over the measurement frame for both MMRs. Every ten frames the excitation amplitude increases in steps of $0.5 \mu\text{T}$.



Supplementary Figure 6: Deviation for larger deflection angles; Experiment 2 - controlled (MMR L). The deviation of the different methods from the reference method ($T_T(\mathcal{P}^{TUSA_F})$) over the maximum deflection angle. For the parameters λ , $\varphi_{\max}$, ω_{nat} , σ_v and σ_u the deviation is quantified using the relative error, for ψ_{start} using the absolute error, and for the alignment of σ_v and σ_u using the cosine similarity.