

Supplementary information

Self-organizing memristive networks as physical learning systems

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Supplementary Information for Self-organizing memristive networks as physical learning systems

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From Circuit Topology to Projector Operators

To build intuition, a memristive network can be viewed as a dynamic plumbing system: voltages act like pressures, currents like fluid flow. Projector operators then serve as mathematical tools that enforce physical constraints, ensuring flow conservation at junctions. They help capture how electrical signals propagate through the network following Kirchhoff's laws, enabling analysis of dynamic information pathways. The projector operator formalism described here for the study of memristive networks is derived from the lumped circuit topology approximation of a memristive network. For a graph-theoretic and more formal introduction to these projectors, see Ref. ^{1,2}.

The projector operators are of the form

$$\Omega_B = B(B^T B)^{-1} B^T, \quad \Omega_A = A(A^T A)^{-1} A^T, \quad (1)$$

where T represents the transpose, B is a circuit's directed incidence matrix and A is its loop matrix, defined below. Projector operators satisfy $\Omega^2 = \Omega$, implying that the eigenspectrum is only 0's and 1's.

For a resistive circuit, each branch j (edge) is a resistor with current i_j and B describes how branches connect to vertices (nodes) in the circuit given an edge orientation:

$$B_{ji} = \begin{cases} -1 & \text{if edge } j \text{ leaves vertex } i, \\ +1 & \text{if edge } j \text{ enters vertex } i, \\ 0 & \text{otherwise.} \end{cases}$$

$B^T \mathbf{i} = 0$ then enforces Kirchhoff's Current Law (KCL) on the current vector $\mathbf{i}$, ensuring current balance at nodes. On the other hand, given arbitrary branches (currents) and mesh

orientations (defined as a closed sequence of branches), Kirchhoff's Voltage Law (KVL) can be written in terms of A , which represents cycles in the graph, with rows corresponding to loops and columns to branches:

$$A_{il} = \begin{cases} +1 & \text{if edge } i \text{ is oriented alike loop } l, \\ -1 & \text{if edge } i \text{ is antioriented as loop } l, \\ 0 & \text{otherwise.} \end{cases}$$

In terms of this matrix, given the vector of voltage drops $\delta \mathbf{v}$, the KVL can be written as $A^T \delta \mathbf{v} = 0$.

Thus, KVL and KCL can be written as:

$$\text{KVL: } \Omega_A \delta \mathbf{v} = 0, \quad \text{KCL: } \Omega_B \mathbf{i} = 0$$

Via cycle and nodal analysis duality arguments (resulting in $BA^T = 0$), it follows that $\Omega_A + \Omega_B = I$, which shows that Ω_A and Ω_B are complementary and project onto cycle and vertex spaces, which can be shown to be equivalent to the voltage and current spaces respectively³.

The dynamics of memristive networks can be written directly in terms of these projectors, making Kirchhoff's laws in the dynamics explicit. For instance, for a graph whose edges are composed of resistors j with value R Ohms, and voltage generators in series v_j Volts, the *network* Ohm's law can be written as¹ $R \mathbf{i} = \Omega_A \mathbf{v}$, where i_j is the solution of the current on the resistor j at equilibrium. The role of the projectors is to "network" and integrate the voltage generators to provide voltage drops on the circuit elements.

A deep link exists between the graph Laplacian⁴ and the projector operators². The (weighted) graph Laplacian is defined as⁵

$$L = B^T G B, \quad (2)$$

where G is a diagonal matrix of edge conductances (note that in standard circuit theory, $G_{jj} = \frac{1}{R_j}$, with R_j the resistance

of edge j). For a connected graph, L has rank $n - 1$ with n nodes. The pseudoinverse L^+ also appears in standard formulas for effective resistances. The quantity most relevant to experiments in two-point conductance measurements is the effective resistance $R_{\text{eff}}(a, b)$ between nodes a and b , given by⁶ $R_{\text{eff}}(a, b) = L_{a,a}^+ + L_{b,b}^+ - L_{a,b}^+ - L_{b,a}^+$. The non-orthogonal projector $\Omega_{B/R^{-1}}$ enforcing current conservation on edges can be expressed in terms of the Laplacian and its pseudoinverse $\Omega_{B/R^{-1}} = B(B^T R^{-1} B)^{-1} B^T R^{-1} = B^T L^+ B G$, where the second equality uses $L = B G B^T$ and $G = R^{-1}$, and where $(\cdot)^+$ denotes the Moore–Penrose pseudoinverse. A short calculation shows that $\Omega_{B/R^{-1}}$ is idempotent ($\Omega_{B/R^{-1}}^2 = \Omega_{B/R^{-1}}$), retaining the defining property of a projector. Moreover, $\Omega_{B/R^{-1}}$ can be related to the orthogonal projector Ω_B (see Ref.³).

Both projector operators (Ω_B , $\Omega_{B/R^{-1}}$) and the graph Laplacian (L^+) encode topological constraints in linear and memristive circuits: L^+ captures node-based current conservation, while the projector explicitly enforces it on edge currents. For further discussion on these connections between projection matrices, the Laplacian, and circuit-theoretic graph operators, and properties of planar circuits, see Refs.^{1,2,7,8}. These formal structures are not only rigorous but also essential for analysing how memory and dynamics emerge collectively from simple local rules, laying the groundwork for understanding SOMNs as learning substrates.

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