nature portfolio

Peer Review File

Challenges in identifying chiral spin textures via the topological Hall effect

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Web links to the author's journal account have been redacted from the decision letters as indicated to maintain confidentiality.

29th Jul 21

Dear Mr Kimbell,

Thank you for submitting your Review manuscript, "The ambiguity between topological and inhomogeneous anomalous Hall effects", to Communications Materials, and please accept our apologies for the long duration of the peer review process, due to absences of reviewers and internal staff during this time of the year.

Your manuscript has now been seen by 3 referees, whose comments are appended below. You will see that while they find your work of potential interest, they have raised substantial concerns that must be addressed. In light of these comments, we cannot accept the manuscript for publication, but are interested in considering a revised version that addresses these serious concerns.

In particular, Reviewer #1 has strong concerns regarding the validity of the model used to claim alternative interpretations of the Hall anomaly, and is asking for more discussion and analysis to support this claim, such as calculations of the net Hall resistivity in heterostructures made of multiple layers/domains with different longitudinal conductivities and different ordinary/anomalous Hall coefficients. Furthermore, Reviewer #2 believes that the transport responses are not sufficient to conclude if the origin of THE is due to chiral spin texture or not. Both Reviewers #2 and #3 are also pointing out that the discussion part of this Review is focused and limited to the SRO system, suggesting therefore to correspondingly refocus the title and introduction to a better targeted audience. Naturally, we would expect all other comments of the referees to be appropriately responded to, including their requests for further information and discussion. However, whether or not you can convincingly respond to these points – and to all other requests – is not something we can assess at the moment.

We hope you will find the referees' comments useful as you decide how to proceed. Should further discussion and analysis allow you to address these criticisms, we would be happy to look at a substantially revised manuscript. However, please bear in mind that we will be reluctant to approach the referees again in the absence of major revisions. If the revision process takes significantly longer than three months, we will be happy to reconsider your paper at a later date, as long as nothing similar has been accepted for publication at Communications Materials or published elsewhere in the meantime.

We are committed to providing a fair and constructive peer-review process. Please don't hesitate to contact us if you wish to discuss the revision in more detail.

When submitting your revised manuscript, please include the following:

-A response letter with a point-by-point reply to each of the referee comments and a description of changes made. Please include the complete referee report in the response letter. Please note that the response letter must be separate to the cover letter to the editors.

-A marked-up version of the manuscript with all changes to the text in a different colored font. Please do not include tracked changes or comments. Please select the file type 'Revised Manuscript - Marked Up' when uploading the manuscript file to our online system.

-A clean version of the manuscript. Please select the file type ‘Article File’.

-An updated Editorial Policy checklist, uploaded as a ‘Related Manuscript File’ type. This checklist is to ensure your paper complies with all relevant editorial policies. If needed, please revise your manuscript in response to these points. Please note that this form is a dynamic ‘smart pdf’ and must therefore be downloaded and completed in Adobe Reader. Clicking this link will download a zip file containing the pdf.

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We understand that due to the current global situation, the time required for revision may be longer than usual. We would appreciate it if you could keep us informed about an estimated timescale for resubmission, to facilitate our planning. Of course, if you are unable to estimate, we are happy to accommodate necessary extensions nevertheless.

Please do not hesitate to contact me if you have any questions or would like to discuss the required revisions further. Thank you for the opportunity to review your work.

Best regards,

Dr Aldo Isidori
Associate Editor
Communications Materials

Reviewers' comments:

Reviewer #1 (Remarks to the Author):

The authors summarized the recent controversy on the topological-Hall-like signature in the oxide heterostructures, especially focusing on SrRuO_{3}. The paper is well organized, clarifying each problem as to the interpretation of Hall anomalies. However, I have a following concern that must be addressed before I could recommend its publication.
The authors strongly highlighted the alternative interpretation of the Hall anomaly by considering the parallel conduction of multiple channels. In their model (for example, Fig. 2), different contributions to Hall resistivity were just superposed as rho_{xy} = sum_{i} rho_{xy}(due to conduction channel #i). However, the parallel conduction of Hall effect is not so simple. Because the superposition principle is not valid for the (Hall) voltage, their model is incorrect. The authors need to clarify this basic and fundamental issue to support their main claim that the coexistence of multi-

AHE mimics the topological-Hall-like response. For example, they need to calculate the net Hall resistivity in the heterostructure made of multiple layers with different longitudinal conductivities and different ordinary/anomalous Hall coefficients without any approximations. Because the authors also generalized that any kind of inhomogeneity can produce topological Hall anomalies, they also need to perform calculations for other cases, for example, the Hall resistivity in a sample with multiple domains with different longitudinal conductivities and different ordinary/anomalous Hall coefficients. The precise analyses and discussions are necessary to support and claim the alternative interpretation. Otherwise, the review would be strongly biased.

Reviewer #2 (Remarks to the Author):

This work addresses the topical problem of topological spin textures, as revealed by magnetotransport measurements. It is now generally believed that when the magnetic order parameter switches along the normal to the film axis, as a function of an applied magnetic field, the skyrmionic textures can appear at intermediate fields, which manifest in transport through the "topological Hall effect" (THE). However, THE provides a signature of chiral magnetic textures and, as a method cannot be used on its own to conclude if the signal is related to skyrmions or not.

Although the work is scientifically sound, it lacks the novelty and impact of a paper suitable for Communications Materials and therefore I do not recommend it for publication. The reasons are the following:

1) Review papers form valuable scientific literature as they summarize the findings of existing literature. So readers can form an idea about the existing knowledge on a topic without having to read all the published works in the field. However, the explanation is very general in the first part of the manuscript that describes skyrmions and Hall effects (first six pages). At the same time, the physics and the origin of the anomalous Hall effect (AHE) and THE have been reported in more detail and depth by the review papers of Nagaoza (Rev. Mod. Phys. 82, 1539 (2010)), Tokura (Chem. Rev. 121, 2857 (2021)), Göbel (Phys. Rep. 895, 1-28 (2021)). Further, the authors focus only on Néel skyrmions (page 2), while a zoo of different chiral skyrmionic phases has been discovered and proposed [Everschor-Sitte (J. Appl. Phys. 124, 240901 (2018), Bogdanov (Nat. Rev. Phys. 2, 492-498 (2020)), Tokura (Chem. Rev. 121, 2857 (2021)), Göbel (Phys. Rep. 895, 1-28 (2021))].

2) I appreciate the author's arguments in the "Skyrmions in thin-film heterostructures" part. There are many examples that momentum and real-space imaging techniques for detecting and observing skyrmionic phases do not overlap. And the transport responses are not sufficient to conclude if the origin of THE is due to chiral spin texture or not.
On the other hand, one should consider that different technique are used for the momentum and real-space observations. Even for the subtraction of the THE, practically two different techniques are needed; magnetotransport and magnetometry. And to direct link and treat the data from transport, one should measure the magnetic and transport properties directly from the same "part" of the sample, and often people use different parts with different dimensions (x-y) of the same sample or magnetotransport on patterned samples and magnetometry on unpatterned samples.
In addition, the authors mention the examples of the Fe3Ge2Te2 [80,81] and antiskyrmions [83,82] that show different behavior in transport and LTEM measurements. However, the samples in both studies have different dimensions; Fe3Ge2Te2: mm size [80] and thin plate of 170 nm [81],

antiskyrmions: 100 nm lamella [83] and bulk material [82]. It was recently shown in anstiskyrmions that the size and the type of the skyrmionic phases strongly depend on the thickness of the samples and protocol of the measurement [Ma (Adv. Mater. 32, 2002043 (2020), Jena (Nat. Commun. 11, 1115 (2020))]. As a result the underlying mechanisms that stabilize skyrmions in systems with different dimensions, such as exchange interactions, DMI, anisotropy, and dipolar interactions change, will affect the size and origin of the THE. Therefore, one should consider many different parameters related to the samples' dimensions, composition and quality, the associated physical properties and the related underlying mechanisms stabilizing skyrmions.

3) The main topic of the review is to discuss the ambiguity between different interpretations of the same Hall data using the SrRuO3 material as a model system (pages 7-21). However, it is limited only to this particular oxide system, and this review is not expanded in other systems. Thus, I believe it is not helpful to other researchers working on the AHE and THE as a probe for chiral spin textures. Further, the manuscript does not provide an adequate critical discussion of the literature, and it only summarizes previous research efforts, and some different sections are not connected. Finally, I find that the last part of the manuscript lacks sufficiently forward-looking and does not provide enough guidance for future research.

Reviewer #3 (Remarks to the Author):

The authors Kimbell et al. have put together a very interesting and thoroughly written review paper. I think the paper is very timely and will resolve common misconceptions in the community. I enjoyed reading it a lot and I think it can be published in Communications Materials after the following remarks have been considered.

1) One thing that I disliked while reading the paper, is the fact that the introduction and the title try to make it sound as if these was a very general discussion of the THE in several material classes. However, the paper, in its discussion, is very restricted to SRO (and maybe related perovskites). I understand that some of the discussed concepts may also be valid for other material classes but they have not been discussed in this paper and are also not under so much discussion in the community, currently.

If I put it in an exaggerated way, the review paper deals with the question whether or not there are skyrmions in SRO. This is a very narrow and well-defined topic. Therefore, I urge the authors to revise the introduction in terms of the materials that are discussed and to change the title to something less general. I think this would also help the paper to better reach the targeted audience.

2) The conclusion was a bit unsatisfying. After a long review paper, I would have hoped for a more definite answer to the question whether or not there are skyrmion in SRO. Based on what was written in the paper, it seems to me that the authors are quite sure that there are skyrmions, even though they may not explain the Hall peak, that was attributed to the THE, alone.

3) Besides these main points, I have some minor comments, as well as references that I feel like should be included in the paper. Since I come from the skyrmion / THE community, most papers are concerned with these topics:

a) Page 2, lines 31-33:
I understand that it is not reasonable to cite papers for all the different types of skyrmion-related textures but this review article could be cited instead:

- Göbel et al. Physics Reports 895, 1 (2021) - Beyond skyrmions: Review and perspectives of alternative magnetic quasiparticles

b) Page 4, line 69 (and also line 86):
I would add that a breaking of certain symmetries by the spin texture is also required for the AHE. Typically, it is the mirror symmetry that is broken by the magnetization in a ferromagnet.

c) Page 6, line 126:
I would like to have a more detailed explanation. How can this effect be topological, when the fluctuations are not characterized by a topological invariant?

d) Page 6, lines 135-137:
That is a very good point, drawing the line between AHE and THE can indeed be somewhat arbitrary. It may be worth citing the following reference that shows that in some systems, the AHE can be transformed into a THE by a coordinate transformation and vice versa:

- Busch et al. Phys. Rev. Research 2, 033112 (2020) - Microscopic origin of the anomalous Hall effect in noncollinear kagome magnets

e) Page 7, lines 167-169:
In the following paper, the THE was also measured at higher temperatures. I think this paper is particularly interesting, because it shows signatures of skyrmions and antiskyrmions by having two THE peaks with opposite signs.

- Sivakumar et al. ACS Nano 14, 13463 (2020) - Topological Hall signatures of two chiral spin textures hosted in a single tetragonal inverse Heusler thin film

f) Page 7, lines 188-193:
In the most recent paper by Raju et al. they show a nice complementary study of MFM and THE measurements and show that for some temperatures the two techniques agree very well, while for other temperatures the THE is much higher than what was expected by the MFM images. The divergence of the THE happens right at the transition between the skyrmion crystal phase and the isolated skyrmion + FM phase. Since SRO might be in a similar transition state, this could also be an explanation for why the THE in SRO is quite large and there has not been a convincing real-space confirmation of the existence of skyrmions, yet. The authors may include a discussion in their paper.

- Raju et al. Nature Communications 12, 2758 (2021) - Colossal topological Hall effect at the transition between isolated and lattice-phase interfacial skyrmions

g) Page 9, lines 306-307
Here the authors could mention the paper from above. Another reference worth citing is the following:

- Denisov et al. PRL 117, 027202 (2016) - Electron Scattering on a Magnetic Skyrmion in the

Nonadiabatic Approximation

h) Page 18, lines 534-542
Sorry, but I do not fully understand this paragraph. Maybe you could explain more precisely how this method helps to find out whether or not we see skyrmions in the sample.

i) Page 19, lines 604-610
Why would it be helpful to have a spatially resolved AHE?

We thank referees 1, 2 and 3 for carefully reading through our manuscript and providing useful feedback which we address point-by-point below.

Reviewer #1:

1. *"The authors summarized the recent controversy on the topological-Hall-like signature in the oxide heterostructures, especially focusing on $SrRuO_3$. The paper is well organized, clarifying each problem as to the interpretation of Hall anomalies. However, I have a following concern that must be addressed before I could recommend its publication."*

   We thank the referee for their positive appraisal of our manuscript and their comments which we address below.

2. *"The authors strongly highlighted the alternative interpretation of the Hall anomaly by considering the parallel conduction of multiple channels. In their model (for example, Fig. 2), different contributions to Hall resistivity were just superposed as rho_{xy} = sum_i rho_{xy} (due to conduction channel #i). However, the parallel conduction of Hall effect is not so simple. Because the superposition principle is not valid for the (Hall) voltage, their model is incorrect. The authors need to clarify this basic and fundamental issue to support their main claim that the coexistence of multi-AHE mimics the topological-Hall-like response. For example, they need to calculate the net Hall resistivity in the heterostructure made of multiple layers with different longitudinal conductivities and different ordinary/anomalous Hall coefficients without any approximations. Because the authors also generalized that any kind of inhomogeneity can produce topological Hall anomalies, they also need to perform calculations for other cases, for example, the Hall resistivity in a sample with multiple domains with different longitudinal conductivities and different ordinary/anomalous Hall coefficients. The precise analyses and discussions are necessary to support and claim the alternative interpretation. Otherwise, the review would be strongly biased."*

   It is correct to observe that the Hall conductivities are additive (for instance $\sigma_{xy}^{tot} = \sigma_{xy}^{i} + \sigma_{xy}^{ii}$ where the indices i and ii denote two channels), while for Hall resistivities (ρ_{xy}) the additivity does not necessarily hold. Below we show that the additivity of ρ_{xy} is only valid for small Hall angles. In the typical measurement geometry, the two conduction channels are electrically in parallel and considered as independent. For a single conduction channel, inversion of the conductivity tensor yields $\rho_{xx} = \frac{\sigma_{xx}}{\sigma_{xx}^2+\sigma_{xy}^2}$ and $\rho_{xy} = \frac{-\sigma_{xy}}{\sigma_{xx}^2+\sigma_{xy}^2}$, which reduces to $\rho_{xx} = \frac{1}{\sigma_{xx}}$ and $\rho_{xy} = \frac{-\sigma_{xy}}{\sigma_{xx}^2}$ for small Hall angles, i.e. $\frac{\sigma_{xy}}{\sigma_{xx}} \ll 1$. In the case of two conduction channels, $\rho_{xx}^{tot} = \frac{1}{\sigma_{xx}^{tot}}$ and

   $$\rho_{xy}^{tot} = \frac{-\sigma_{xy}^{tot}}{(\sigma_{xx}^{tot})^2} = \frac{-(\sigma_{xy}^{i} + \sigma_{xy}^{ii})}{(\sigma_{xx}^{tot})^2} = \frac{-\sigma_{xy}^{i}}{(\sigma_{xx}^{tot})^2} + \frac{-\sigma_{xy}^{ii}}{(\sigma_{xx}^{tot})^2} .$$

   Assuming the small angle hypothesis, $\sigma_{xy} = -\sigma_{xx}^2\rho_{xy}$ and we obtain

   $$\rho_{xy}^{tot} = \left(\frac{\sigma_{xx}^{i}}{\sigma_{xx}^{tot}}\right)^2 \rho_{xy}^{i} + \left(\frac{\sigma_{xx}^{ii}}{\sigma_{xx}^{tot}}\right)^2 \rho_{xy}^{ii} ,$$

   which shows that ρ_{xy}^{tot} can be decomposed into a linear combination of ρ_{xy}^{i} and ρ_{xy}^{ii}, weighted by the relative effective conductivity of each channel. The effective conductivity depends on volume fraction, geometry, and intrinsic conductivity of the material in each channel. For $SrRuO_3$ thin films the ratio between the longitudinal and Hall conductivity is typically $\frac{\sigma_{xy}}{\sigma_{xx}} \sim 10^{-3}$, hence the addition of Hall resistivities, measured as transverse voltages, is valid.

The decomposition of the total Hall resistivity into a sum of components is used throughout the literature in various materials, for example see DOIs: 10.1088/2515-7639/ab7a03, 10.1038/s41567-018-0307-5, 10.1038/s41467-018-08041-9, 10.1038/s41565-017-0044-4.

We have added a brief explanation of this to the Hall effects section.

Reviewer #2:

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1. *"This work addresses the topical problem of topological spin textures, as revealed by magnetotransport measurements. It is now generally believed that when the magnetic order parameter switches along the normal to the film axis, as a function of an applied magnetic field, the skyrmionic textures can appear at intermediate fields, which manifest in transport through the "topological Hall effect" (THE). However, THE provides a signature of chiral magnetic textures and, as a method cannot be used on its own to conclude if the signal is related to skyrmions or not.*

   Although the work is scientifically sound, it lacks the novelty and impact of a paper suitable for Communications Materials and therefore I do not recommend it for publication. The reasons are the following:"

   We thank the referee for their critical review of our manuscript and pointing out several important issues which we address point-by-point below.

   The main point of our review is not that the THE cannot determine which chiral spin texture is present, but rather that the THE is not sufficient evidence to determine whether there is a chiral spin texture at all, which as you say is generally believed. As far as we are aware, there are no review papers on this topic. This topic is important and timely as there are many papers being published in this area with disagreeing interpretations of results, and there is a lot of confusion and misconceptions in the community.

2. *"Review papers form valuable scientific literature as they summarize the findings of existing literature. So readers can form an idea about the existing knowledge on a topic without having to read all the published works in the field. However, the explanation is very general in the first part of the manuscript that describes skyrmions and Hall effects (first six pages). At the same time, the physics and the origin of the anomalous Hall effect (AHE) and THE have been reported in more detail and depth by the review papers of Nagaoza (Rev. Mod. Phys. 82, 1539 (2010)), Tokura (Chem. Rev. 121, 2857 (2021)), Göbel (Phys. Rep. 895, 1-28 (2021)). Further, the authors focus only on Néel skyrmions (page 2), while a zoo of different chiral skyrmionic phases has been discovered and proposed [Everschor-Sitte (J. Appl. Phys. 124, 240901 (2018), Bogdanov (Nat. Rev. Phys. 2, 492-498 (2020)), Tokura (Chem. Rev. 121, 2857 (2021)), Göbel (Phys. Rep. 895, 1-28 (2021))]."*

   We do not attempt to give a complete overview of the theory of the AHE or THE. We included a general summary of Néel skyrmions and Hall effects only to give readers the relevant background information in order to understand the problem in question, i.e. the problem of measuring spin chirality in thin films using Hall effects.

Our discussion applies generally to any chiral spin textures. We highlight Néel-type as this is the most commonly expected chiral spin-texture in thin films. However, we agree we focus too much on Néel-skyrmions specifically and this may be misleading. We have removed the section for skyrmions and incorporated a broader discussion of skyrmions and other skyrmionic phases into the introduction. We have expanded the section on Hall effects as we highlight some important points central to this review.

We also direct the reader to these more in-depth reviews for further reading and thank the referee for providing additional useful references. However, none of these reviews discuss inhomogeneities in the anomalous Hall effect, which is where the novelty lies in our work.

3. *"I appreciate the author's arguments in the "Skyrmions in thin-film heterostructures" part. There are many examples that momentum and real-space imaging techniques for detecting and observing skyrmionic phases do not overlap. And the transport responses are not sufficient to conclude if the origin of THE is due to chiral spin texture or not.*

   On the other hand, one should consider that different technique are used for the momentum and real-space observations. Even for the subtraction of the THE, practically two different techniques are needed; magnetotransport and magnetometry. And to direct link and treat the data from transport, one should measure the magnetic and transport properties directly from the same "part" of the sample, and often people use different parts with different dimensions (x-y) of the same sample or magnetotransport on patterned samples and magnetometry on unpatterned samples."

   A good point and similar to the one we make in lines 315-320 in the original manuscript:

   > "Furthermore, this subtraction technique requires that the sample has the same properties in both measurements, and both measurements are made with the same magnetic field calibration and field sweep-rate […] This is further complicated by the fact that films are often patterned into Hall bars after magnetometry measurements, which changes the region being sampled in measurements and could change the film properties."

   However, we admit this point was somewhat hidden, and think it is a very important point that deserves stressing further. We have moved this point to its own subsection 'Subtracting two measurements' within the 'Problems with the topological interpretation' section.

4. *"In addition, the authors mention the examples of the Fe3Ge2Te2 [80,81] and antiskyrmions [83,82] that show different behavior in transport and LTEM measurements. However, the samples in both studies have different dimensions; Fe3Ge2Te2: mm size [80] and thin plate of 170 nm [81], antiskyrmions: 100 nm lamella [83] and bulk material [82]. It was recently shown in anstiskyrmions that the size and the type of the skyrmionic phases strongly depend on the thickness of the samples and protocol of the measurement [Ma (Adv. Mater. 32, 2002043 (2020), Jena (Nat. Commun. 11, 1115 (2020))]. As a result the underlying mechanisms that stabilize skyrmions in systems with different dimensions, such as exchange interactions, DMI, anisotropy, and dipolar interactions change, will affect the size and origin of the THE. Therefore, one should consider many different parameters related to the samples' dimensions, composition and quality, the associated physical properties and the related underlying mechanisms stabilizing skyrmions."*

Agreed. We mention in the review "It is possible that milling samples into thin lamellae affects the stability of skyrmions" – but we were not precise. We do not intend to imply that the inconsistencies mean the interpretations are invalid, only that it is very challenging to correlate these different measurements. Resolving this problem by measuring LTEM and Hall effects on a simultaneously on a single lamella is an ongoing research effort in some spintronics groups, but the equipment needed is very uncommon. In the new version we clarify this and add additional discussion in the section "Skyrmions in thin-film heterostructures".

5. *"The main topic of the review is to discuss the ambiguity between different interpretations of the same Hall data using the SrRuO3 material as a model system (pages 7-21). However, it is limited only to this particular oxide system, and this review is not expanded in other systems. Thus, I believe it is not helpful to other researchers working on the AHE and THE as a probe for chiral spin textures."*

   We believe there is a large enough body of work around $SrRuO_3$ that it is appropriate to focus the review on this topical material. Whilst some of the points are specific to $SrRuO_3$, many of the criticisms apply generally to any system hosting an intrinsic anomalous Hall effect. To give a few examples: interface modification has been identified as mimicking the topological Hall effect in topological insulators (DOI: 10.1103/PhysRevX.10.011012), thickness variations may account for topological Hall effects reported in Heusler alloys (DOI: 10.1038/s41598-017-13211-8), and the effect of inhomogeneity in the anomalous Hall effect is ignored in recent studies claiming to directly observe topological Hall effects in Co/Pt (DOI: 10.1103/PhysRevB.103.054429).

   We agree we have not made connections to other material systems clearly enough, and we ideally hope researchers can apply the lessons learnt in $SrRuO_3$ to other systems. We have therefore added additional some additional discussions and references linking these criticisms to other relevant material systems throughout the review.

6. *"Further, the manuscript does not provide an adequate critical discussion of the literature, and it only summarizes previous research efforts, and some different sections are not connected."*

   We politely disagree with this point, but the perceived lack of critical discussion may be a side-effect of the structure of our review. A review summarises key findings in the literature, whereas a perspective gives a more critical discussion. In this manuscript we attempt to do both in that order. We initially summarise existing work in the sections "The topological Hall effect in $SrRuO_3$" and "The inhomogeneous anomalous Hall effect in $SrRuO_3$". Here we try to remain unbiased and summarise the important points of literature without much critical discussion. We then critically discuss the literature more broadly in the sections "Problems with topological interpretation" and "Problems with inhomogeneity interpretation".

   We hope that sections follow a logical order, however we agree that some sections were out of place. We have removed the section "structural modifications" as this is essentially the same as "asymmetric interfaces", and the contents of the section "Magnetic inhomogeneity from temperature dependence" has been moved to the section "Problems with topological interpretation" as it seems unconnected in its current location.

7. *"Finally, I find that the last part of the manuscript lacks sufficiently forward-looking and does not provide enough guidance for future research."*

We have expanded on and added additional experimental suggestions to the future work section and changed the associated Figure. We hope that these suggestions provide useful ideas and starting points for other researchers.

We agree that the original conclusion was rather weak, and we have now strengthened and expanded upon our stance in the conclusion. We try to highlight that based on recent literature, without convincing supplementary measurements, the measurement of a THE should not be considered sufficient evidence for any chiral spin texture in a film. Further measurements should focus on separating Hall effect contributions by tuning the magnetic texture with some control parameter, or by imaging chiral spin textures more directly with spin polarised scanning tunnelling microscopy.

Reviewer #3:

1. *"The authors Kimbell et al. have put together a very interesting and thoroughly written review paper. I think the paper is very timely and will resolve common misconceptions in the community. I enjoyed reading it a lot and I think it can be published in Communications Materials after the following remarks have been considered."*

   We thank the referee for their positive appraisal of our manuscript, and for providing valuable criticisms which we address point-by-point below.

2. *"One thing that I disliked while reading the paper, is the fact that the introduction and the title try to make it sound as if these was a very general discussion of the THE in several material classes. However, the paper, in its discussion, is very restricted to SRO (and maybe related perovskites). I understand that some of the discussed concepts may also be valid for other material classes but they have not been discussed in this paper and are also not under so much discussion in the community, currently.*
   If I put it in an exaggerated way, the review paper deals with the question whether or not there are skyrmions in SRO. This is a very narrow and well-defined topic. Therefore, I urge the authors to revise the introduction in terms of the materials that are discussed and to change the title to something less general. I think this would also help the paper to better reach the targeted audience."

   Good point – the review is essentially a case study of $SrRuO_3$. However, we believe there is a large enough body of experimental work being done with this material to justify this focus. Although some of the issues are specific to $SrRuO_3$, many of the highlighted criticisms apply universally to materials hosting an intrinsic anomalous Hall effect. For example, interface modification has been identified as mimicking the topological Hall effect in topological insulators (DOI: 10.1103/PhysRevX.10.011012), thickness variations may account for topological Hall effects reported in Heusler alloys (DOI: 10.1038/s41598-017-13211-8), and the effect of inhomogeneity in the anomalous Hall effect is ignored in recent studies claiming to directly observe topological Hall effects in Co/Pt (DOI: 10.1103/PhysRevB.103.054429).

   In the original manuscript we did not make the connections to other material systems clearly. We have therefore added additional references and discussions linking these criticisms to other relevant material systems throughout the review.

3. *“The conclusion was a bit unsatisfying. After a long review paper, I would have hoped for a more definite answer to the question whether or not there are skyrmion in SRO. Based on what was written in the paper, it seems to me that the authors are quite sure that there are skyrmions, even though they may not explain the Hall peak, that was attributed to the THE, alone.”*

   It is difficult to give a definite answer about skyrmions in SRO as there is no consensus in the literature, but we believe this makes the topic more interesting and important to cover in a review article.

   We believe it is unlikely that skyrmions exist in all of the structures investigated in the literature. One of the main points of the review is that the evidence for skyrmions in many materials systems is far from clear due to materials and technical issues – highlighting these issues is important to the community of scientists working on this highly topical area and will lead to scientific progress.

   We do, however, concede that our original conclusion was weak. We have now strengthened and expanded upon our stance in the conclusion. We have tried to highlight that based on recent literature, without convincing supplementary measurements, the measurement of a THE should not be considered sufficient evidence for any chiral spin texture in a film.

4. *“Besides these main points, I have some minor comments, as well as references that I feel like should be included in the paper. Since I come from the skyrmion / THE community, most papers are concerned with these topics:*

   Page 2, lines 31-33:
   I understand that it is not reasonable to cite papers for all the different types of skyrmion-related textures but this review article could be cited instead: Göbel et al. Physics Reports 895, 1 (2021) - Beyond skyrmions: Review and perspectives of alternative magnetic quasiparticles”

   Thank you for notifying us of this recent review. We have removed our section on skyrmions and instead added a broader discussion of skyrmions in the introduction including a reference to this review.

5. *“Page 4, line 69 (and also line 86):*
   I would add that a breaking of certain symmetries by the spin texture is also required for the AHE. Typically, it is the mirror symmetry that is broken by the magnetization in a ferromagnet.”

   Agreed – breaking time-reversal symmetry is required for a finite AHE. This was implied by considering a ferromagnetic film, but specifying that it is required would be more clear and more complete. We have changed this appropriately.

6. *“Page 6, line 126:*
   I would like to have a more detailed explanation. How can this effect be topological, when the fluctuations are not characterized by a topological invariant?”

   This is a very important point in the discussion, and we now discuss this in detail in the new subsection “Is the topological Hall effect really topological”. To which the answer is: not necessarily, according to the most common definition.

One can consider a triad of spins which have a scalar spin chirality, $\chi_{ijk} = S_i \cdot (S_j \times S_k)$. This spin chirality gives a finite Berry phase, which acts as an effective field and contributes to the Hall response. However, this spin chirality is not necessarily topologically nontrivial.

Hall effects from topologically trivial spin chirality have been discussed in relation to chiral excitations in manganites near T_C (DOIs: 10.1103/PhysRevB.57.10248, 10.1103/PhysRevLett.84.757), as well as chiral ground states such as in frustrated ferromagnets (DOI: 10.1126/science.1058161). In these cases the effect was referred to as an "AHE", but in other works "THE" is used to refer to any Hall effect due to chiral spin textures, including topologically trivial textures (DOIs: 10.1038/s41563-019-0454-9, 10.1103/PhysRevLett.108.156601, 10.1073/pnas.2023588118). For example, related to the referee's #9 comment, the Nature Communications paper also claim fluctuations enhances THE, and we believe the fluctuation there is not "truly" topological either.

The confusion is whether the term "THE" includes Hall effects due to chiral spin textures (either static or dynamic) that are not topological, i.e. not characterized by integer Chern number. Most papers on the subject state that spin chirality gives a Berry phase which gives a Hall response which is called the THE. However, this definition doesn't require the spin texture to be topological. Experimentally, the defining feature of THE is the hump/dip feature in Hall effect which vanishes in the high field limit, but this cannot distinguish whether the origin is topological or not.

In practise, "THE" is used to include Hall effects from topologically trivial spin textures, and it is also how we use the term in our review.

It is a difficult situation, we don't think "topological Hall effect" is a very appropriate name, and "chiral Hall effect" has already been coined for another contribution, and "geometrical Hall effect" could be confused with k-space geometry. We decided that attempting to change terminology now will probably lead to more confusion so we have stuck to the common definition of a THE.

We have clarified these points in the new subsection.

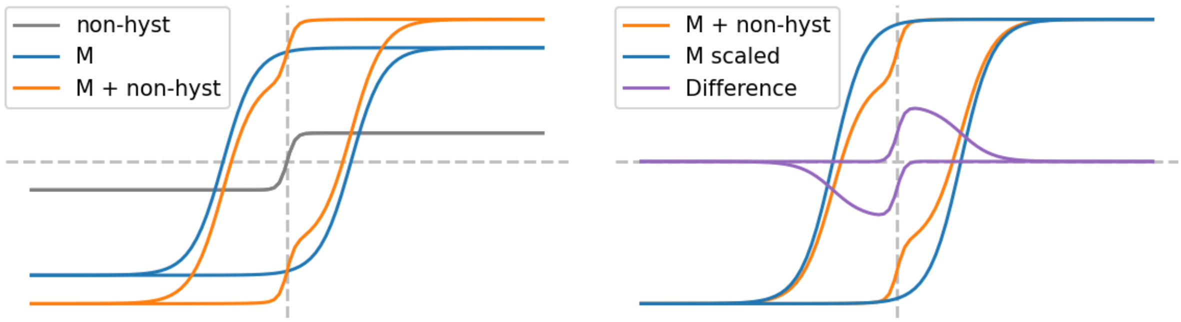
7. *"Page 6, lines 135-137:*
   That is a very good point, drawing the line between AHE and THE can indeed be somewhat arbitrary. It may be worth citing the following reference that shows that in some systems, the AHE can be transformed into a THE by a coordinate transformation and vice versa: Busch et al. Phys. Rev. Research 2, 033112 (2020) - Microscopic origin of the anomalous Hall effect in noncollinear kagome magnets"

   Thank you, this is an interesting and relevant paper which we have added to the discussion mentioned in the previous point.

8. *"Page 7, lines 167-169:*
   In the following paper, the THE was also measured at higher temperatures. I think this paper is particularly interesting, because it shows signatures of skyrmions and antiskyrmions by having two THE peaks with opposite signs. Sivakumar et al. ACS Nano 14, 13463 (2020) -

Topological Hall signatures of two chiral spin textures hosted in a single tetragonal inverse Heusler thin film"

Thank you, this is an interesting paper. This material has many similarities with $SrRuO_3$, such as an AHE which changes sign with temperature with positive peaks in the Hall effect appearing near this sign change. Many of the same criticisms we highlight in our review apply here – it is assumed that the AHE is proportional to the magnetisation, and two measurements made in two different systems are subtracted to give the THE. The magnetisation in particular needs to be corrected for residual fields and effects from the substrate/holder, which have a ferromagnet-like signal shown in the supplementary info. The authors state that because the holder is non-hysteretic and the THE is hysteretic, that it does not qualitatively affect any conclusions. However, this is not generally true. If we consider two equivalent hysteresis loops, add a non-hysteretic contribution to one, then scale the two to match saturation, we get a hysteretic difference between the two loops. i.e. an incorrect non-hysteretic correction to magnetometry gives an apparent THE. This is shown schematically below:



This is not to say that the analysis in the manuscript is incorrect, but it is an example of details that are often overlooked in the literature and are something we are trying to highlight with this review.

9. *Page 7, lines 188-193:*
   In the most recent paper by Raju et al. they show a nice complementary study of MFM and THE measurements and show that for some temperatures the two techniques agree very well, while for other temperatures the THE is much higher than what was expected by the MFM images. The divergence of the THE happens right at the transition between the skyrmion crystal phase and the isolated skyrmion + FM phase. Since SRO might be in a similar transition state, this could also be an explanation for why the THE in SRO is quite large and there has not been a convincing real-space confirmation of the existence of skyrmions, yet. The authors may include a discussion in their paper.

   - Raju et al. Nature Communications 12, 2758 (2021) - Colossal topological Hall effect at the transition between isolated and lattice-phase interfacial skyrmions

   Thank you for notifying us of this paper. The results here are very important for this problem, and directly addresses one of the major concerns with the topological interpretation. We have added a discussion of this work to the section "the inconsistency between topological Hall effect and real-space imaging", which this paper directly addresses.

10. *"Page 9, lines 306-307*
    Here the authors could mention the paper from above. Another reference worth citing is the

following: Denisov et al. PRL 117, 027202 (2016) - Electron Scattering on a Magnetic Skyrmion in the Nonadiabatic Approximation"

A relevant reference which has been added to the discussion along with the reference from the previous point.

11. *"Page 18, lines 534-542*
    Sorry, but I do not fully understand this paragraph. Maybe you could explain more precisely how this method helps to find out whether or not we see skyrmions in the sample."

    This is a good point, we include $M(T)$ as a more sensitive way of observing magnetic inhomogeneity, but by itself doesn't say anything about chirality and it is out of place here. We have moved the contents of this section to the subsection "No steps in magnetisation" in "Problems with topological interpretation" as it is more relevant there.

12. *"Page 19, lines 604-610*
    Why would it be helpful to have a spatially resolved AHE?"

    The problem in question is whether the Hall peaks are due to a spatial inhomogeneity in the AHE. If there was some way to directly measure a spatially resolved AHE, one could directly and unambiguously show whether this is the case. The only way to get some amount of spatial resolution in Hall measurements is to make a Hall bar with lots of transverse contacts, and this could reveal whether Hall anomalies are due to an inhomogeneous AHE if the inhomogeneity occurs over a large scale. We have rephrased this part of the review to try to make this point clearer.

16th Nov 21

Dear Mr Kimbell,

Thank you for submitting your revised Review manuscript, "The ambiguity between topological and inhomogeneous anomalous Hall effects", to Communications Materials. It has now been seen again by our 3 referees, whose comments are appended below. You will see that while Reviewers #2 and #3 are now happy with the revisions and recommend the publication of your manuscript, Reviewer #1 has still a few important technical concerns. In particular, Reviewer #1 is still concerned by the superposition model of Hall resistivity [resistance], which according to the reviewer should take into account also the thickness of each layer. Furthermore, the reviewer is still not convinced of the generalized applicability of the parallel conductance model.

We are still very much interested in the possibility of publishing your Review in Communications Materials, but we would like to consider your response to Reviewer #1's concerns in the form of a revised manuscript before we make a decision on publication. We therefore invite you to revise and resubmit your manuscript, taking into account the points raised.

We are committed to providing a fair and constructive peer-review process. Please don't hesitate to contact us if you wish to discuss the revision in more detail.

When submitting your revised manuscript, please include the following:

-A response letter with a point-by-point reply to each of the referee comments and a description of changes made. Please include the complete referee report in the response letter. Please note that the response letter must be separate to the cover letter to the editors.

-A marked-up version of the manuscript with all changes to the text in a different colored font. Please do not include tracked changes or comments. Please select the file type ‘Revised Manuscript - Marked Up’ when uploading the manuscript file to our online system.

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- Other unique identifiers (such as DOIs and hyperlinks for any other datasets)
- At a minimum, a statement confirming that all relevant data are available from the authors
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DATA SOURCES: We strongly encourage authors to deposit all new data associated with the paper in a persistent repository where they can be freely and enduringly accessed. We recommend submitting the data to discipline-specific, community-recognized repositories, where possible and a list of recommended repositories is provided at http://www.nature.com/sdata/policies/repositories.

If a community resource is unavailable, data can be submitted to generalist repositories such as figshare or Dryad Digital Repository. Please provide a unique identifier for the data (for example a DOI or a permanent URL) in the data availability statement, if possible. If the repository does not provide identifiers, we encourage authors to supply the search terms that will return the data. For data that have been obtained from publically available sources, please provide a URL and the specific data product name in the data availability statement. Data with a DOI should be further cited in the methods reference section.

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We hope to receive your revised paper within three months; please let us know if you aren’t able to submit it within this time so that we can discuss how best to proceed. If we don’t hear from you, and the revision process takes significantly longer, we will close your file. In this event, we will still be happy to reconsider your paper at a later date, as long as nothing similar has been accepted for publication at Communications Materials or published elsewhere in the meantime.

We understand that due to the current global situation, the time required for revision may be longer than usual. We would appreciate it if you could keep us informed about an estimated timescale for resubmission, to facilitate our planning. Of course, if you are unable to estimate, we are happy to accommodate necessary extensions nevertheless.

Please do not hesitate to contact me if you have any questions or would like to discuss these revisions further. We look forward to seeing the revised manuscript and thank you for the opportunity to review your work.

Best regards,

Dr Aldo Isidori
Associate Editor
Communications Materials

Reviewers' comments:

Reviewer #1 (Remarks to the Author):

The authors added a comment on Hall effect in the case of the parallel conduction (Interface modification in Fig. 1c). Their formula for Hall resistivity rho_{xy}^{tot} = (sigma_{xx}^i / sigma_{xx}^{tot})^2 rho_{xy}^i + (sigma_{xx}^ii / sigma_{xx}^{tot})^2 rho_{xy}^ii (equation 1) is valid if rho and sigma correspond to "resistance" and "conductance", not "resistivity" and "conductivity". Therefore, the superposition model of Hall resistivity needs to be further corrected not only by the longitudinal conductivity but also by the size of each layer (l: length, w: width, t: thickness). When these corrections are taken into account, does the model of "multiple AHE interpretation" still fit to the experimental results? The author may comment on this issue by using/estimating the correction parameters "weight of longitudinal conductivity" and "sample size".

On the other hand, I am still not convinced of the authors' generalized interpretation that the above parallel conductance model can be applicable to the other cases such as "Thickness variation" and "Defects" in Fig. 1c. In those cases, the conduction properties need to be described by using the series-circuit model, not by using parallel-circuit model of equation 1. If the authors would not like to change their generalized model for describing Hall effect in any kind of inhomogeneity, further arguments need to be included.

The above issues, which I also pointed out in the last process, remain to be clarified. I can recommend the publication after the authors address the above.

Reviewer #2 (Remarks to the Author):

The points raised in the previous round of review have been satisfactorily addressed by the authors. I believe that they did a great work, the review manuscript is now well organized and clear, and therefore I recommend it for publication in Commun. Materials.

Reviewer #3 (Remarks to the Author):

I kindly thank the authors for their reply to my report. It was very interesting to read their responses and I agree to their arguments. Furthermore, I think they have appropriately implemented the suggested changes.

The only small dissatisfaction is that they have not altered the title. However, due to the improved discussion, this problem is not severe anymore which is why I can respect and accept this decision of the authors.

For this reason, my positive first impression has been reconfirmed and I renew my suggestion to publish this nice paper in Communications Materials.

We thank the reviewers for their time and positive appraisal of our manuscript. Below, we address the points from reviewer #1, who raised valid criticisms of the simplification of decomposing a Hall resistivity into a linear sum of Hall resistivities from individual components.

Remarks to reviewer #1

The three limiting geometries of inhomogeneous films consisting of two components are shown in Figure 1. We show that, if the Hall angle and magnetoresistance is small in both materials, the total Hall resistivity can be treated as a linear combination of the individual materials' Hall resistivity. Any arbitrary geometry can be constructed by combining these elements, so in principle any arbitrary geometry can also be described as a linear combination of Hall resistivities.

Homogeneous material

For a quasi-2D material the conductivity and resistivity tensors are written

$$\mathcal{S} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} , \tag{1}$$

$$\mathcal{R} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix} , \tag{2}$$

with $\mathcal{S} = \mathcal{R}^{-1}$. If we assume our material is isotropic then these tensors reduce to:

$$\mathcal{S} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix} , \tag{3}$$

$$\mathcal{R} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{xx} \end{pmatrix} . \tag{4}$$

If we assume that the Hall angle is small, i.e. $\rho_{xy} \ll \rho_{xx}$ and $\sigma_{xy} \ll \sigma_{xx}$, then inverting the resistivity tensor gives:

$$\sigma_{xx} = \frac{\rho_{xx}}{\rho_{xx}^2 + \rho_{xy}^2} \approx \frac{1}{\rho_{xx}} , \tag{5}$$

$$\sigma_{xy} = \frac{-\rho_{xy}}{\rho_{xx}^2 + \rho_{xy}^2} \approx \frac{-\rho_{xy}}{\rho_{xx}^2} . \tag{6}$$

In $SrRuO_3$ at low temperature, typically $\rho_{xx} \sim 10^3 |\rho_{xy}|$, so the small Hall angle approximation is valid.

For a single slab of material the equations for current and electric field are:

$$E_x = \rho_{xx} J_x + \rho_{xy} J_y , \tag{7}$$

$$E_y = \rho_{yx} J_x + \rho_{yy} J_y , \tag{8}$$

$$J_x = \sigma_{xx} E_x + \sigma_{xy} E_y , \tag{9}$$

$$J_y = \sigma_{yx} E_x + \sigma_{yy} E_y . \tag{10}$$

We apply a current J_y, in the steady state we assume $J_x = 0$, the electric field in the y direction is much larger than in the x direction, and we remember that our material is isotropic. The equations then reduce to:

$$E_x = \rho_{xy} J_y , \tag{11}$$

$$E_y = \rho_{xx} J_y , \tag{12}$$

$$J_x = 0 , \tag{13}$$

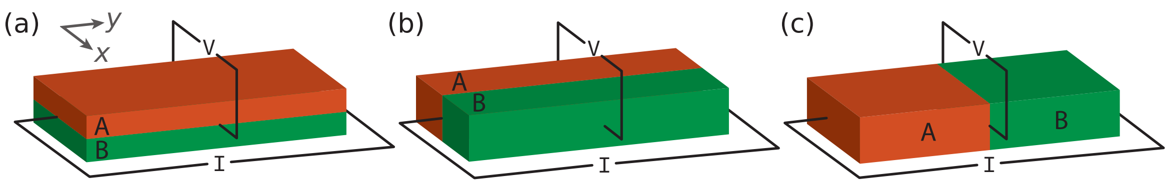

Figure 1: Three structures under consideration, consisting of different materials A and B: (a) stacked parallel, (b) adjacent parallel, and (c) series. The current is applied along the y direction and a transverse voltage is measured along x.

Structures (a) and (b)

In structures (a) and (b) in Figure 1, the longitudinal current is in parallel. The overall current density is the average current density of the layers weighted by their volume fraction, f:

$$J_y^{\mathrm{tot}} = f^A J_y^A + f^B J_y^B \ , \tag{14}$$

and the longitudinal voltage/electric field across the device is the same in both layers, so:

$$E_y^A = E_y^B \ , \tag{15}$$

$$\rho_{xx}^A J_y^A = \rho_{xx}^B J_y^B \ . \tag{16}$$

From this we find the current density in each material:

$$J_y^A = \frac{\rho_{xx}^B}{\rho_{xx}^A} J_y^B \ , \tag{17}$$

$$J_y^{\mathrm{tot}} = f^A \frac{\rho_{xx}^B}{\rho_{xx}^A} J_y^B + f^B J_y^B \ , \tag{18}$$

$$J_y^{\mathrm{tot}} = \left(\frac{f^A \rho_{xx}^B + f^B \rho_{xx}^A}{\rho_{xx}^A} \right) J_y^B \ , \tag{19}$$

$$J_y^B = \left(\frac{\rho_{xx}^A}{f^A \rho_{xx}^B + f^B \rho_{xx}^A} \right) J_y^{\mathrm{tot}} \ , \tag{20}$$

and similarly

$$J_y^A = \left(\frac{\rho_{xx}^B}{f^A \rho_{xx}^B + f^B \rho_{xx}^A} \right) J_y^{\mathrm{tot}} \ , \tag{21}$$

which is the result for two resistors in parallel.

The measured transverse field in both (a) and (b) is the weighted average of the transverse field generated in each layer, giving:

$$E_x^{\mathrm{tot}} = f^A E_x^A + f^B E_x^B \ . \tag{22}$$

The transverse electric field generated in each material is proportional to the current density through that material, so:

$$E_x^{\mathrm{tot}} = f^A \rho_{xy}^A J_y^A + f^B \rho_{xy}^B J_y^B \ . \tag{23}$$

Subbing Equations (20) and (21) into (23), we find

$$E_x^{\mathrm{tot}} = f^A \rho_{xy}^A \left(\frac{\rho_{xx}^B}{f^B \rho_{xx}^A + f^A \rho_{xx}^B} \right) J_y^{\mathrm{tot}} + f^B \rho_{xy}^B \left(\frac{\rho_{xx}^A}{f^A \rho_{xx}^B + f^B \rho_{xx}^A} \right) J_y^{\mathrm{tot}} \ , \tag{24}$$

$$E_x^{\mathrm{tot}} / J_y^{\mathrm{tot}} = \rho_{xy}^{\mathrm{tot}} = \frac{f^A \rho_{xx}^B \rho_{xy}^A + f^B \rho_{xx}^A \rho_{xy}^B}{f^B \rho_{xx}^A + f^A \rho_{xx}^B} \ , \tag{25}$$

$$\rho_{xy}^{\mathrm{tot}} = \alpha \rho_{xy}^A + \beta \rho_{xy}^B \ , \tag{26}$$

where α and β are constants if ρ_{xx} is constant. Measurements are usually made as a function of magnetic field so we assume the relative change in resistivity due to magnetoresistnace is small compared to the relative change in Hall resistivity. In $SrRuO_3$ the magnetoresistance is $\sim 2\%$ at 2 T, so this assumption is reasonable. Thus the total measured Hall resistivity, ρ_{xy}^{tot}, in these two structures is a linear combination of the Hall resistivities of the two constituent materials, weighted by a function of longitudinal resistance and volume fraction.

Structure (c)

In structure (c) in Figure 1, the materials are longitudinally in series, so the current density through both is the same:

$$J_y^{\mathrm{tot}} = J_y^A = J_y^B \ , \tag{27}$$

The transverse electric field generated in each material is proportional to the current density:

$$E_x^A = \rho_{xy}^A J_y^{\mathrm{tot}} \ , \tag{28}$$

$$E_x^B = \rho_{xy}^B J_y^{\text{tot}} \ . \tag{29}$$

If the two regions A and B are macroscopic then there could be spatial dependence of the measured overall transverse field versus the distance y. Here, we assume the regions are sufficiently small that the measured overall transverse field is a weighted average of the field generated by each region:

$$E_x^{\text{tot}} = f^A E_x^A + f^B E_x^B = f^A \rho_{xy}^A J_y^{\text{tot}} + f^B \rho_{xy}^B J_y^{\text{tot}} \ . \tag{30}$$

$$E_x^{\text{tot}} / J_y^{\text{tot}} = \rho_{xy}^{\text{tot}} = f^A \rho_{xy}^A + f^B \rho_{xy}^B \ , \tag{31}$$

which is again a linear combination of the individual Hall resistivities. In this case there is no dependence on the relative longitudinal resistivities.

Arbitrary geometries – an example

These equations can be easily expanded to any number of layers, and can be combined to form any arbitrary geometry in principle, although the equations can quickly become unwieldy with this approach. We show a simple example below of how a grid of pillars of material B in a matrix of material A can be modelled, illustrated in Figure 2. The structure can be modelled by two parallel regions, one of material A, and one of material A and B in series, which we treat as the composite material C.

For the parallel bars of material A and composite material C, shown in Figure 2c, we use Equation (25):

$$\rho_{xy}^{\text{tot}} = \frac{f_p^A \rho_{xx}^C \rho_{xy}^A + f_p^C \rho_{xx}^A \rho_{xy}^C}{f_p^C \rho_{xx}^A + f_p^A \rho_{xx}^C} \ , \tag{32}$$

where f_p^A and f_p^C are the volume fraction of A and C from the parallel configuration Figure 2c.

For the composite material C we have from Equation (31) and resistors in series:

$$\rho_{xy}^{\text{C}} = f_s^A \rho_{xy}^A + f_s^B \rho_{xy}^B \ , \tag{33}$$

$$\rho_{xx}^{\text{C}} = f_s^A \rho_{xx}^A + f_s^B \rho_{xx}^B \ , \tag{34}$$

where f_s^A and f_s^B is the volume fraction of A and B in series within the composite material C shown in Figure 2d. Subbing this into the previous equation we get the total Hall resistivity

$$\rho_{xy}^{\text{tot}} = \frac{f_p^A (f_s^A \rho_{xx}^A + f_s^B \rho_{xx}^B) \rho_{xy}^A + f_p^C \rho_{xx}^A (f_s^A \rho_{xy}^A + f_s^B \rho_{xy}^B)}{f_p^C \rho_{xx}^A + f_p^A (f_s^A \rho_{xx}^A + f_s^B \rho_{xx}^B)} \ . \tag{35}$$

This can be rearranged to

$$\rho_{xy}^{\text{tot}} = \frac{(f_p^A f_s^A \rho_{xx}^A + f_p^A f_s^B \rho_{xx}^B + f_p^C f_s^A \rho_{xx}^A) \rho_{xy}^A + (f_p^C f_s^B \rho_{xx}^A) \rho_{xy}^B}{f_p^C \rho_{xx}^A + f_p^A (f_s^A \rho_{xx}^A + f_s^B \rho_{xx}^B)} \ , \tag{36}$$

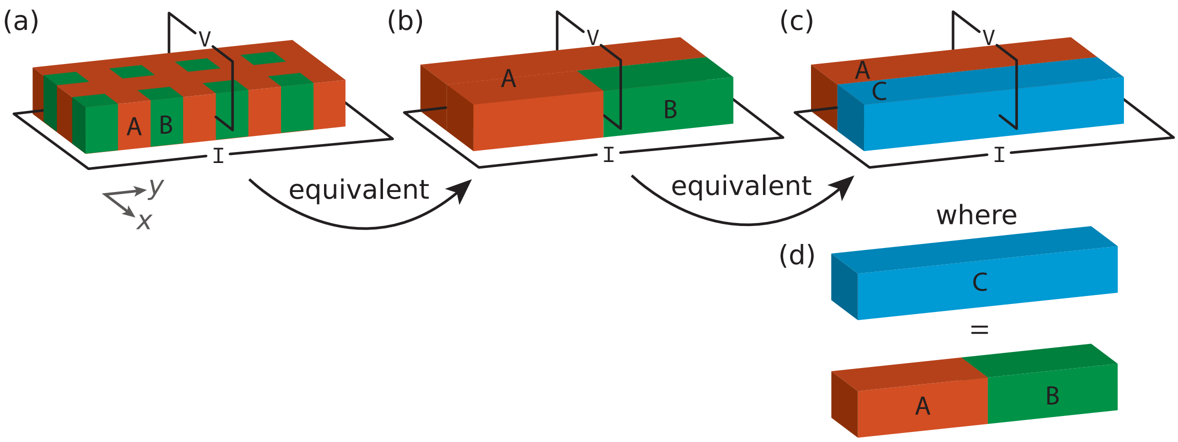

Figure 2: Modelling the Hall resistivity of a sample consisting of B pillars in an A matrix. We assume the transverse voltage measurement is made over the whole length of the sample and averaged. Ignoring interface resistances this structure can be simplified into two parallel bars, one consisting only of A and the other consisting of A and B in series. The A and B in series can be treated as the composite material C.

again reducing to a linear combination of the two constituent Hall resistivties:

$$\rho_{xy}^{\text{tot}} = \alpha' \rho_{xy}^{A} + \beta' \rho_{xy}^{B} \,, \tag{37}$$

where α' and β' are constants.

Comparing models to data

There is a significant challenge in quantitatively comparing these predictions of Hall resistivity to data because many constants must first be known accurately. One must know the longitudinal resistance of each individual material, their volume fractions, and their geometry. To our knowledge no one has yet made quantitative comparisons between models and measurements. Perhaps the clearest example of the linear summation to date is from Wang *et. al* [1], where the sums of a 4 UC and 5 UC Hall resistance matches reasonably to the measured Hall resistivity of intermediate thicknesses.

We have attempted to compare the models above to this data by fixing ρ_{xx} and f of the 4 and 5 UC regions to values determined in the paper [1]. The results are shown in Figure 3, where the series model shows reasonable agreement to the data. This model is the simple summation of Hall resistances by volume fraction. The model in the original paper used the average Hall resistance weighted by the fraction of 4 to 5 UC regions (i.e. ignoring thickness), which gave a similarly reasonable fit to the data. In both cases, the peaks expected by the model are larger than those seen in practise. This is probably because we assume that the two magnetisation loops of the 4 and 5 UC regions act independently in the non-integer thickness samples. In reality a coupling between the two regions is expected, which may increase the coercive field of the 4 UC component and decrease the coercive field of the 5 UC component, resulting in smaller peaks than the prediction for decoupled regions.

In general, the the decomposition of a total Hall resistivity into a linear combination of constituent Hall resistivities is a convenient simplification which is often used in the literature for $SrRuO_3$ [1–6] and other materials such as magnetic topological insulators [7–9].

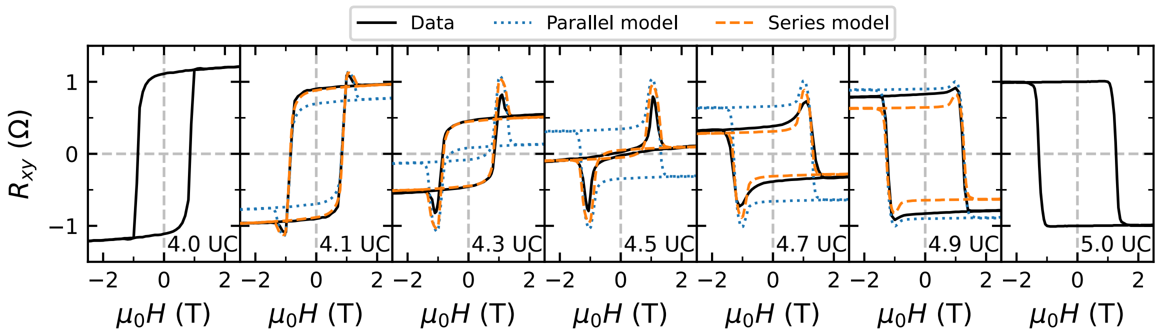

Figure 3: Comparison of parallel and series models of Hall resistivity with measured data in $SrRuO_3$ films with thickness inhomogeneity. Films between 4 and 5 UC thick comprise of a mixture of 4 and 5 UC regions, and the total Hall resisitivity is modelled as a weighted average of the two individual Hall resistivities. The series model appears to give reasonable agreement with the data. Data taken from Wang *et al.* [1].

References

[1] Lingfei Wang, Qiyuan Feng, Han Gyeol Lee, Eun Kyo Ko, Qingyou Lu, and Tae Won Noh. Controllable Thickness Inhomogeneity and Berry Curvature Engineering of Anomalous Hall Effect in $SrRuO_3$ Ultrathin Films. *Nano Letters*, 20(4):2468–2477, April 2020.

[2] T C van Thiel, D J Groenendijk, and A D Caviglia. Extraordinary Hall balance in ultrathin $SrRuO_3$ bilayers. *Journal of Physics: Materials*, 3(2):025005, March 2020.

[3] Liang Wu, Fangdi Wen, Yixing Fu, Justin H. Wilson, Xiaoran Liu, Yujun Zhang, Denis M. Vasiukov, Mikhail S. Kareev, J. H. Pixley, and Jak Chakhalian. Berry phase manipulation in ultrathin $SrRuO_3$ films. *Physical Review B*, 102(22):220406, December 2020.

[4] D. J. Groenendijk, C. Autieri, T. C. van Thiel, W. Brzezicki, J. R. Hortensius, D. Afanasiev, N. Gauquelin, P. Barone, K. H. W. van den Bos, S. van Aert, J. Verbeeck, A. Filippetti, S. Picozzi, M. Cuoco, and A. D. Caviglia. Berry phase engineering at oxide interfaces. *Physical Review Research*, 2(2):023404, June 2020.

[5] Graham Kimbell, Paul M. Sass, Bart Woltjes, Eun Kyo Ko, Tae Won Noh, Weida Wu, and Jason W. A. Robinson. Two-channel anomalous Hall effect in $SrRuO_3$. *Physical Review Materials*, 4(5):054414, May 2020.

[6] Dongxing Zheng, Yue-Wen Fang, Senfu Zhang, Peng Li, Yan Wen, Bin Fang, Xin He, Yan Li, Chenhui Zhang, Wenyi Tong, Wenbo Mi, Haili Bai, Husam N. Alshareef, Zi Qiang Qiu, and Xixiang Zhang. Berry Phase Engineering in $SrRuO_3/SrIrO_3/SrTiO_3$ Superlattices Induced by Band Structure Reconstruction. *ACS Nano*, 15(3):5086–5095, February 2021.

[7] K. M. Fijalkowski, M. Hartl, M. Winnerlein, P. Mandal, S. Schreyeck, K. Brunner, C. Gould, and L. W. Molenkamp. Coexistence of Surface and Bulk Ferromagnetism Mimics Skyrmion Hall Effect in a Topological Insulator. *Physical Review X*, 10(1):011012, January 2020.

[8] Bin Jia, Shuai Zhang, Zhe Ying, Hangkai Xie, Bo Chen, Muhammad Naveed, Fucong Fei, Minhao Zhang, Danfeng Pan, and Fengqi Song. Unconventional anomalous Hall effect in magnetic topological insulator $MnBi_4Te_7$ device. *Applied Physics Letters*, 118(8):083101, February 2021.

[9] Fei Wang, Xuepeng Wang, Yi-Fan Zhao, Di Xiao, Ling-Jie Zhou, Wei Liu, Zhidong Zhang, Weiwei Zhao, Moses H. W. Chan, Nitin Samarth, Chaoxing Liu, Haijun Zhang, and Cui-Zu Chang. Interface-induced sign reversal of the anomalous Hall effect in magnetic topological insulator heterostructures. *Nature Communications*, 12(1):79, January 2021.

25th Jan 22

Dear Mr Kimbell,

Your Review titled "The ambiguity between topological and inhomogeneous anomalous Hall effects" has now been seen again by Reviewer #1, whose comments appear below. In the light of this and the other two referees' advice, I am delighted to say that we are happy, in principle, to publish it in Communications Materials under a Creative Commons 'CC BY' open access license.

We therefore invite you to revise your paper one last time to comply with our journal policies and formatting style in order to maximise the accessibility and therefore the impact of your work.

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Best regards,

Dr Aldo Isidori
Associate Editor
Communications Materials

REVIEWERS' COMMENTS:

Reviewer #1 (Remarks to the Author):

I appreciate the efforts made by the authors to address the points made in my reports. The authors have clarified the ambiguity in their discussion on the Hall resistivity in the multi-domain states. The manuscript will contribute to the deep understanding of the topological magnetism and related transport properties. Now I support the publication from Communications Materials.

Reviewer #1:

"I appreciate the efforts made by the authors to address the points made in my reports. The authors have clarified the ambiguity in their discussion on the Hall resistivity in the multi-domain states. The manuscript will contribute to the deep understanding of the topological magnetism and related transport properties. Now I support the publication from Communications Materials."

We thank the reviewer for their careful scrutiny of our work and the positive appraisal of our manuscript.