

Supplementary Materials for “Scalable Sondheimer oscillations driven by commensurability between two quantizations”

Note 1. Materials and Methods.

Crystal growth. Cd single crystals were grown using the vapor-phase transport method. A quartz tube containing appropriate amounts of starting material, Cd (99.9997%), was kept at the growth temperature for two weeks in a two-zone furnace. Shiny plate-like crystals were produced under a temperature gradient of 300 – 200°C.

Fabrication of the samples. The production method followed previous work [1]. First, the crystallographic orientation of the bulk crystal was determined using X-ray diffraction. Subsequently, the crystal was securely affixed to an SEM stub in the desired orientation. The next critical step was to employ a focused beam of Ga ions to meticulously carve a rectangular slab, a lamella, from the crystal. This process unfolds in three stages.

Initially, a high current of 50 nA was used to create two gaps within the crystal (Fig.S1a). These gaps were separated by a section of the crystal, approximately $\sim 25 \mu\text{m}$ thick and $150 \mu\text{m}$ long. Following this, a smaller current of 15 nA was applied to smoothen the crystal section, resulting in moderately flat sidewalls. The lamella remained attached to the parent crystal through two beams at its top.

Afterwards, using a precise 7 nA current, both sides of the lamella were cut, ensuring parallel sidewalls and a high level of smoothness. The obtained flakes were then carefully transferred to an alumina substrate. It is crucial to note that a very thin layer of “Araldite” epoxy adhesive must be applied to the substrate beforehand. Once positioned atop the epoxy adhesive, capillary forces naturally shaped the epoxy around the lamella, extending smoothly to each of its top surfaces without covering them.

After allowing the epoxy adhesive to dry naturally, the substrate was taken to a sputtering machine, depositing a 300 nm layer of Au on the lamella. To further refine the device, the Scios 2 system was utilized. Initially, the Au layer covering the active part of the device was removed with an acceleration voltage of 5 kV and an ion current of 2 nA. Subsequently, the overall device shape, including the contact positions, was precisely cut out at 0.3 nA and 30 kV. Finally, the Au layer away from the device was severed, ensuring a current flow exclusively through the device. With these steps completed, the device became ready for measurement (Fig.S1 b and c).

Samples with a thickness of approximately 12.6 and 25 μm are shown in Fig. S1 d and e. We used a focused ion beam to etch lamellae with sufficient size to measure resistance using the four-electrode method, as shown in

Fig.S1 f and g.

Note 2. Estimation of the mean-free-path in Cd single crystals.

Cadmium is a compensated metal with a carrier density of $n = p = 3.48 \times 10^{22} \text{ cm}^{-3}$ [2]. The free-electron Drude conductivity is $\sigma = \frac{2e^2}{3\pi\hbar}(k_{Fe}^2 + k_{Fh}^2)\ell_0$. Assuming $k_{Fe} = k_{Fh} = (3\pi^2n)^{\frac{1}{3}}$, one finds that $\rho_0\ell = 11.5 \times 10^{-16} \Omega\cdot\text{m}^2$, which is close to what was deduced previously by Gall [3]. This is the expression that we used to extract ℓ_0 from ρ_0 for each sample in table I. As seen in the table, the extracted value is $\approx 100 \mu\text{m}$ in the thickest sample, and remains comparable to the thickness with thinning.

Note 3. Magneto-morphic oscillations in samples with a thickness between 97.5 μm to 475 μm .

Fig.S2 shows magneto-morphic oscillations of longitudinal and transverse electrical resistivity in the three thickest cadmium single crystals at 2 K. Sondheimer oscillations are clearly visible in the Hall signal, where the background is smaller due to the compensation between electrons and holes. By inverting the measured resistivity tensor, we quantified the conductivity tensor using $\sigma_{ij} = \frac{-\rho_{ij}}{\rho_{ii}^2 + \rho_{ij}^2}$.

As shown in Fig.S3 a, b, and c, when $B < B_1$, the oscillating longitudinal conductivity scales as $\propto B^{-2.5}e^{-B/3\Delta B}$. When $B > B_1$, its amplitude follows B^{-4} . The same result is obtained for the oscillating Hall conductivity, as shown in Fig.S3 d, e, and f.

Note 4. The field and thickness dependence of oscillations in the two regimes

As mentioned above, B_1 is the threshold field separating the two types of field-dependent amplitude. When $B < B_1$, multiplying the amplitude of oscillations by d^2 and plotting $\delta\sigma_{ij}B^{2.5}d^2$ yields the results shown in Fig. S5a and b, in which all curves are on top of each other.

On the other hand, when the field exceeds B_1 , the amplitude falls off as $\propto B^{-4}$. Multiplying the amplitude of oscillations by d^3 and plotting $\delta\sigma_{ij}G_0^{-1}\ell_B^{-8}d^3$ leads to Fig. S5c and d. One can see that the scaling is not as good as in the case of low-field ($B < B_1$) oscillations. Nevertheless, d^{-3} gives a much better scaling than d^{-2} and d^{-4} , indicating that the thickness dependence in this regime is close to d^{-3} .

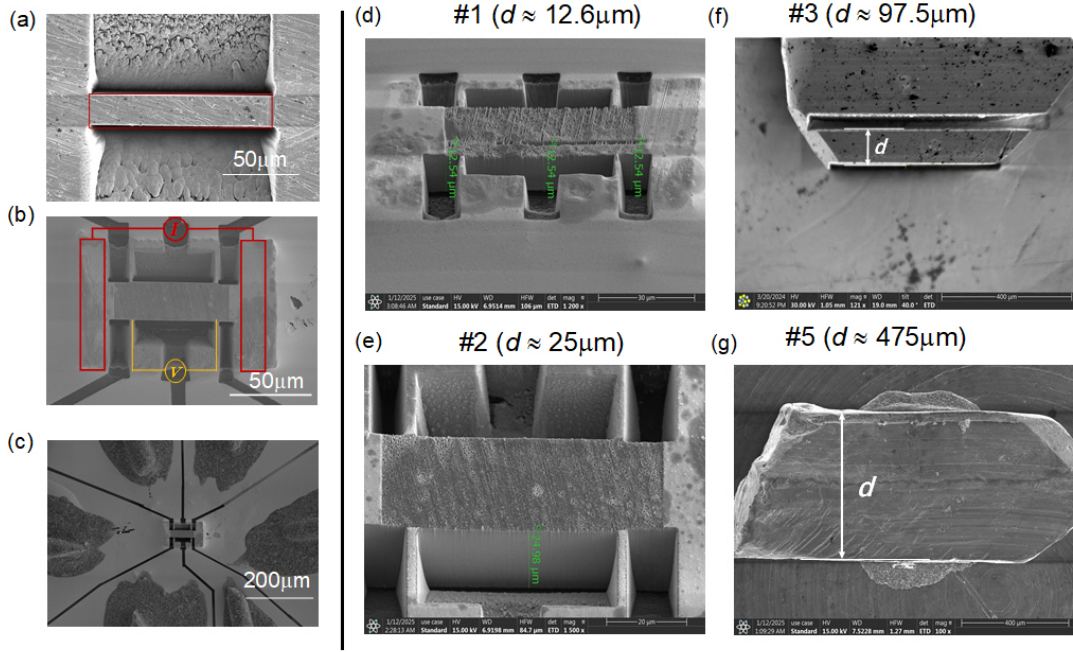


FIG. S1. **Sample preparation and thickness measurement.** (a) Scanning electron-beam microscopy (SEM) image of the sample (marked by red box) etched by FIB. The scale bar denotes $50 \mu\text{m}$. (b) Ion beam etching was employed to fabricate electrodes for the samples. The current electrode is highlighted with the red box, and voltage electrodes are marked by yellow line. (c) Attach the conductive electrode to the gold wire with silver gel. Thickness measurements were conducted on samples of $12.6 \mu\text{m}$ (d), $25 \mu\text{m}$ (e), $97.5 \mu\text{m}$ (f) and $475 \mu\text{m}$ (g).

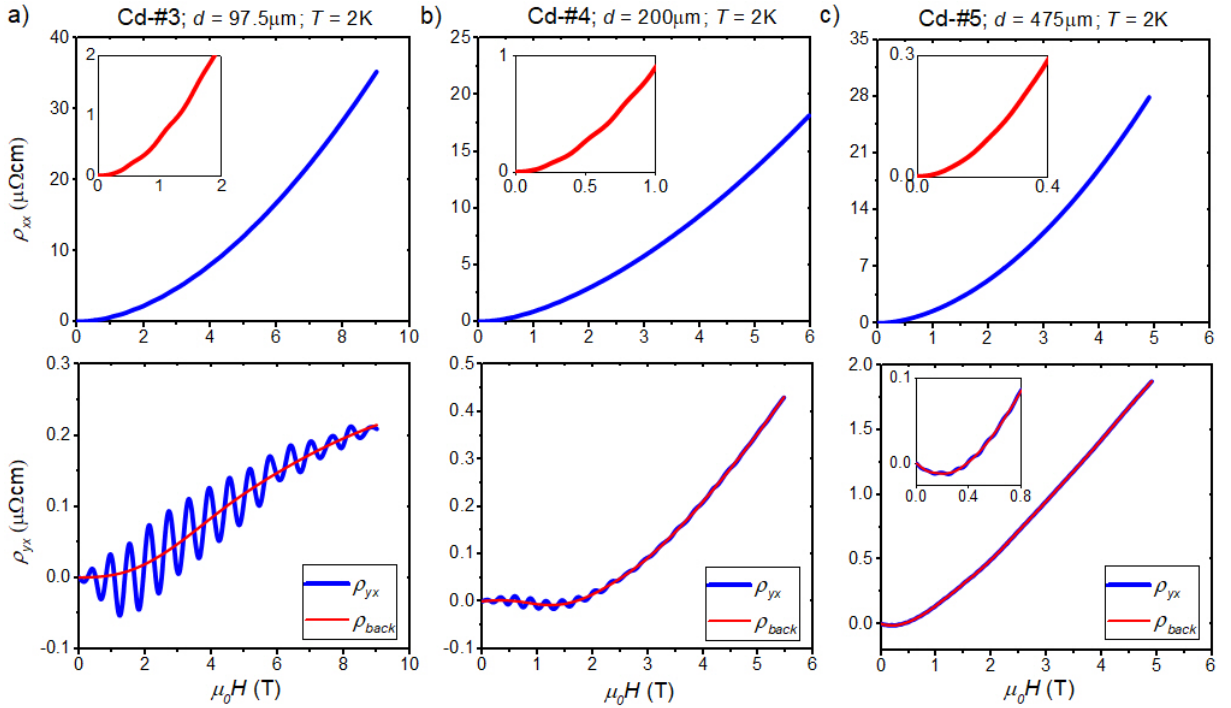


FIG. S2. **Longitudinal and transverse resistivity in the thicker samples.** Longitudinal and transverse resistivity at 2 K in the $97.5 \mu\text{m}$ (a), in the $200 \mu\text{m}$ (b) and in the $475 \mu\text{m}$ (c) samples.

Note 5. Magneto-morphic oscillations in copper samples.

single crystals at 2 K. Inverting the measured resistiv-

Fig.S6 shows magneto-morphic oscillations in longitudinal and transverse electrical resistivity for copper

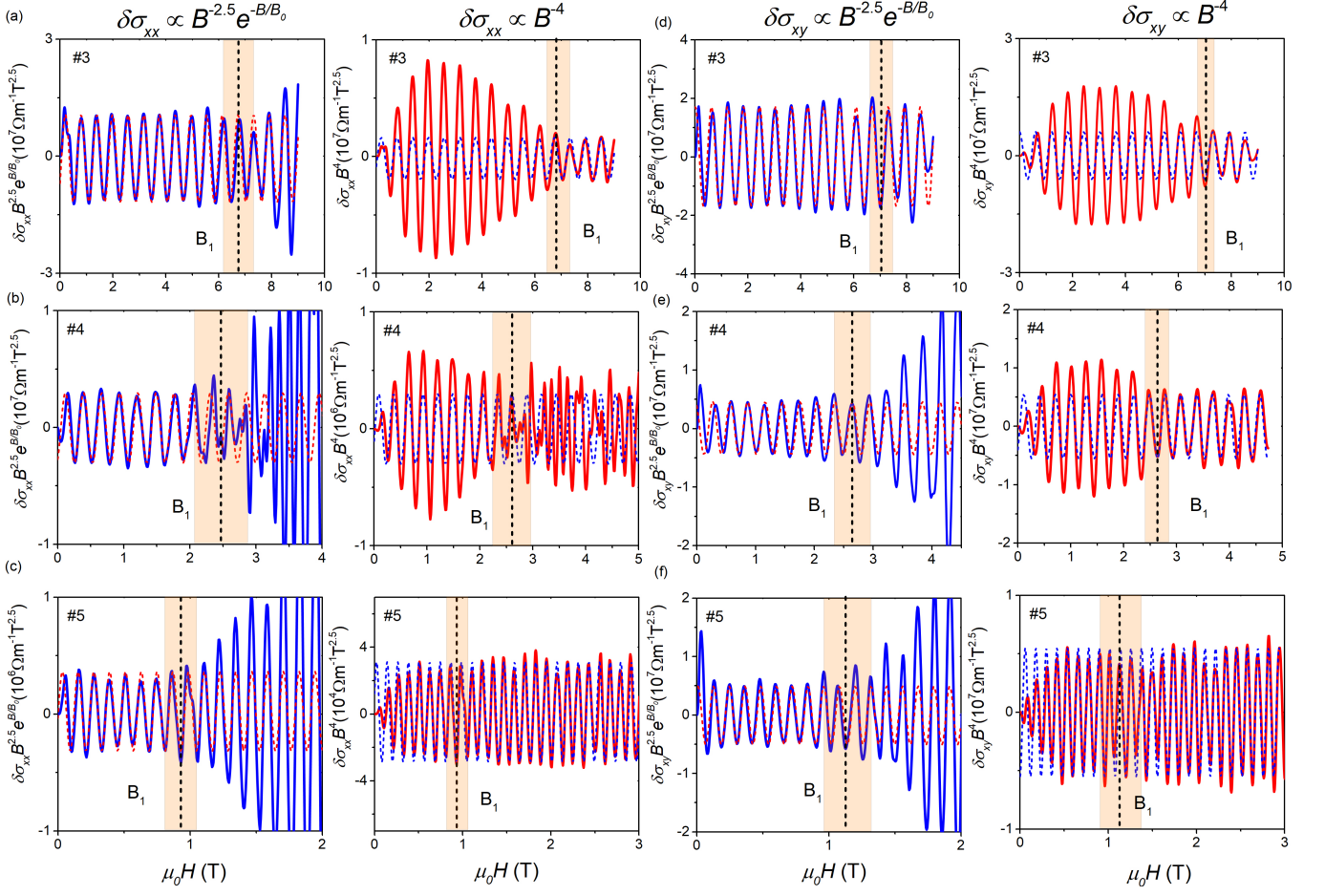


FIG. S3. **Field dependence of the amplitude of oscillations in the thicker samples.** (a) The oscillating longitudinal conductivity, $\delta\sigma_{xx}$ multiplied by $B^{2.5}e^{B/B_0}$ (left) and by B^4 (right) in the 97.5 μm sample. The dashed line indicates the sine fitting. The yellow rectangular bar indicates the error space for B_1 . (b) Same for the 200 μm sample. (c) Same for the 475 μm sample. (d), (e) and (f) Same for oscillating Hall conductivity $\delta\sigma_{xy}$. In all cases, below B_1 , the first multiplication yields a flat amplitude but not the second. Above B_1 , it is the inverse.

ity tensor, we quantified the conductivity tensor using $\sigma_{ij} = \frac{-\rho_{ij}}{\rho_{ii}^2 + \rho_{jj}^2}$. The non-oscillatory background is obtained through polynomial fitting (red line in Fig. S6c). After subtracting this background, the oscillating component was extracted. We found that multiplying the extracted signal by $B^{2.5}$ yields an almost constant amplitude for the first few oscillations, as shown in Fig. S6d (top panel).

Consistent with this observation, a similar scaling ($\propto B^{2.5}$) is observed in the Hall conductivity of the same crystal, as shown in Fig. S6d (bottom panel). Applying the same method to the 200 μm sample, the field dependence for both the longitudinal conductivity and transverse conductivity was found to scale as $\propto B^{2.5}$, as shown in Fig. S6e.

Fig. S6f shown the oscillation period as a function of thickness d . The product of the oscillation period and thickness, $\Delta B \times d$, is equal to $14.7 \times 10^{-6} \text{ T} \cdot \text{m}$, which is same with previous report in copper crystal[4].

Note 6. Oscillations of dimensionless conductivity as a function of dimensionless period.

As mentioned in the main text, we can rewrite the empirical expression 4 using Ω as: $\delta\sigma^{pp} = \alpha G_0 k_{Fz} k_{Fr} \ell_B \Omega^{-2} e^{-\Omega/\pi}$, where α is a dimensionless fitting parameter.

A sine fit, $\alpha \sin(\Omega - \phi)$, is applied to the data of the 12.6, 25, 97.5, 200 and 475 μm samples, marked by a solid blue line in the main text Fig. 9. The fitting amplitude α are shown in Fig. S7a, revealing that for longitudinal conductivity, $\alpha \approx 1$, while for transverse conductivity, α is approximately twice the longitudinal value, as shown in Fig. S7b. Fig. S7c show the phase in longitudinal and transverse conductivity, respectively. The phase difference, $\phi_{xy} - \phi_{xx}$, is close to $-\frac{\pi}{2}$, as shown in Fig. S7d.

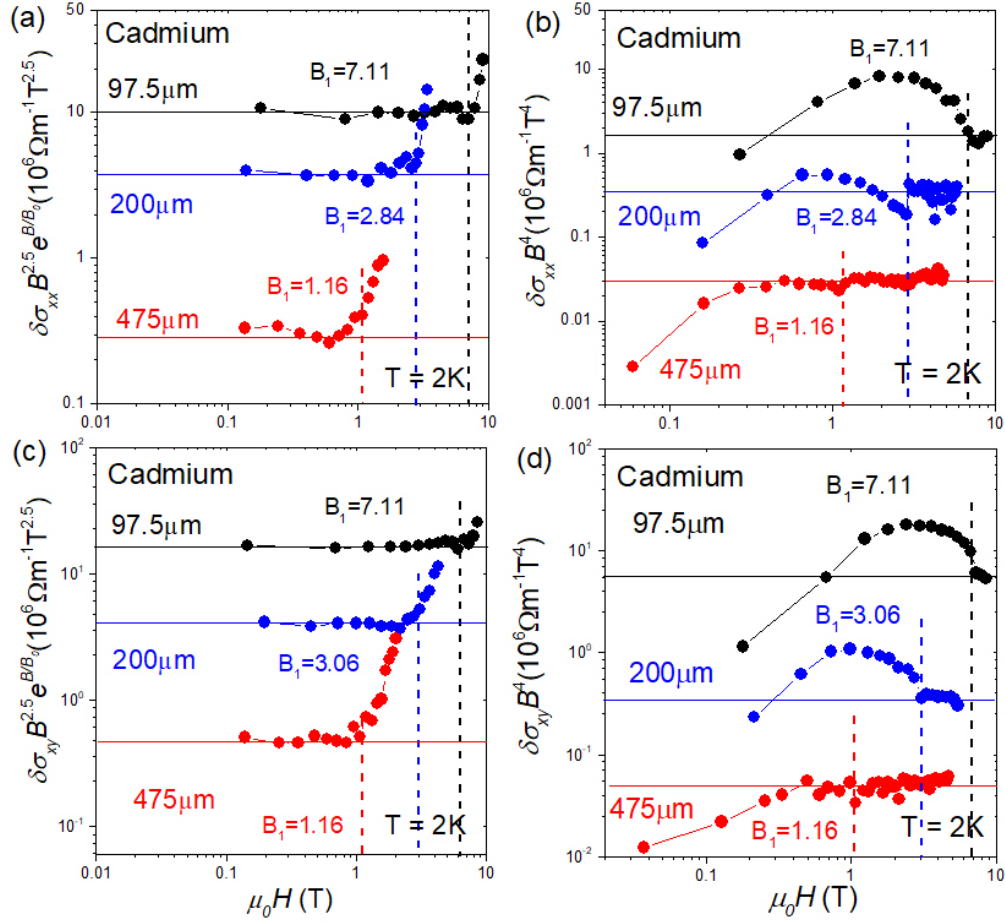


FIG. S4. **The field dependence of oscillation amplitude in two regimes.** (a) Longitudinal conductivity multiplied by $\propto B^{2.5}e^{B/B_0}$ as function of magnetic field. (b) Longitudinal conductivity multiplied by $\propto B^4$ as function of magnetic field. The flatness in two different field windows implies that when the field is less than B_1 , the SO oscillation can be expressed as $\delta\sigma_{xx} \propto B^{-2.5}e^{-B/B_0}$, but when the field is larger than B_1 , it follows B^{-4} . (c), (d) Same for Hall conductivity.

Note 7. Derivation of energy distance between two neighboring levels l and $l+1$.

The main text shows (Fig.8b) that the confinement of electrons in real space modifies the smooth Fermi surface and induces a stepped structure in k -space. The energy separation δE between two neighboring levels, l and $l+1$, can be expressed as:

$$\begin{aligned} \delta E &= \frac{\partial E}{\partial k_z} \Delta k_z \\ &= \frac{\hbar^2}{2m^*} \frac{\partial k_z^2}{\partial k_z} k_d \\ &= \frac{\hbar^2}{2\pi m^*} k_d \frac{\partial A}{\partial k_z} \end{aligned} \quad (S1)$$

This result corresponds to Eq. 7 in the main text. Furthermore, for a given region of phase space, the degeneracy is defined as the number of quantum states it contains, which is proportional to the ratio of the region's volume to the volume of a single quantum state. There-

fore, the degeneracy within the ring of area $k_d \frac{\partial A}{\partial k_z}$ is:

$$D_z = \frac{k_d \frac{\partial A}{\partial k_z}}{(2\pi)^2} = (2\pi)^{-2} k_d \frac{\partial A}{\partial k_z} \quad (S2)$$

In Eq.12, the tunneling action is: $s = eBk_d^{-1}(\frac{\partial A}{\partial k_z})^{-1}$. As mentioned above, in momentum space, $k_d \frac{\partial A}{\partial k_z}$ represents the area of each carved ring between two neighboring levels, indexed l and $l+1$. Thus, the $(k_d \frac{\partial A}{\partial k_z})^{-1}$ is an area in real space. Thus, s in Eq.12 is a magnetic flux times the elementary charge e .

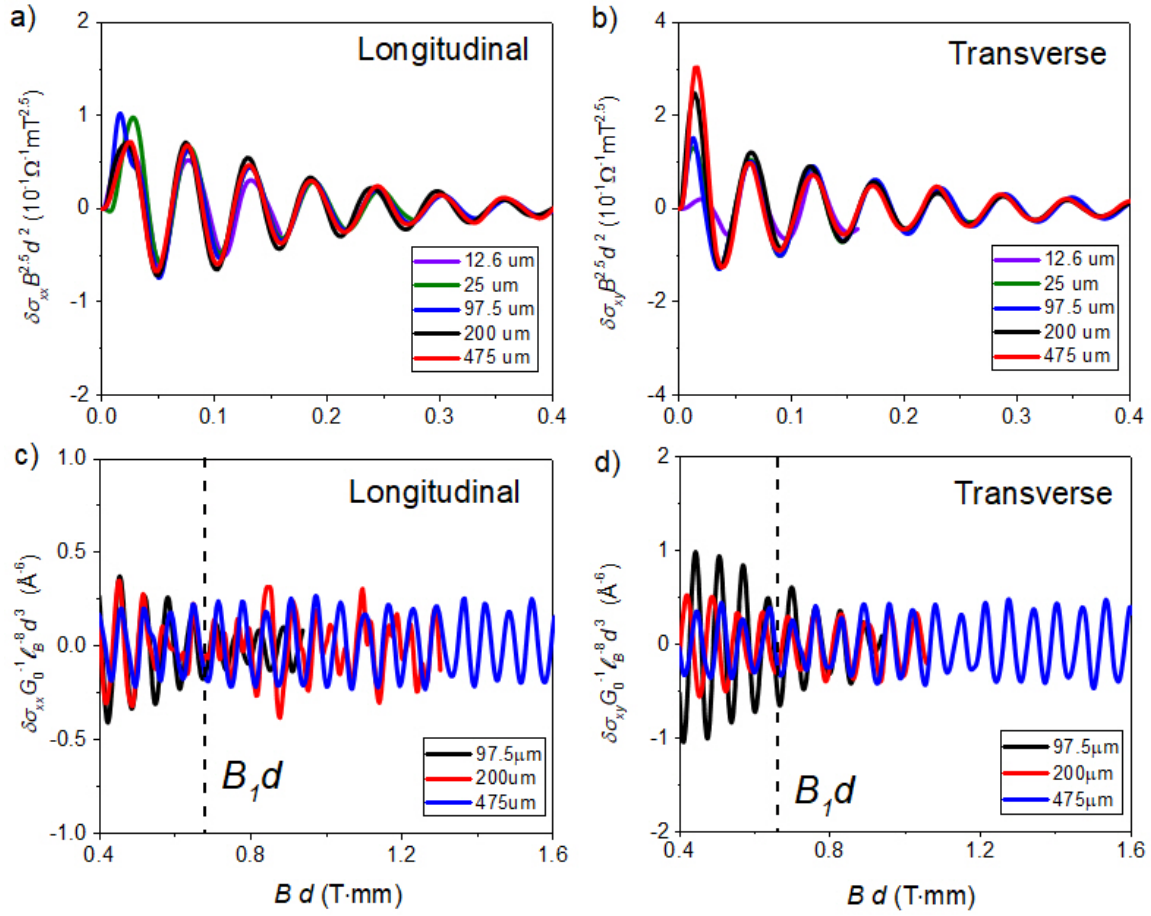


FIG. S5. **Scaling of the oscillations with thickness.** When $B < B_1$, (a) the amplitude of the σ_{xx} oscillation multiplied by d^2 is plotted as a function of the product of the field and thickness. All curves collapse. (b) Same procedure for the Hall conductivity σ_{xy} . When $B > B_1$, (c) the vertical axis is the conductivity divided by the quantum of conductance G_0 and multiplied by $\ell_B^{-8} \times d^3$. The horizontal axis, linear in magnetic field, represents the dimensionless product: Bd . (d) Same for the Hall conductivity σ_{xy} .

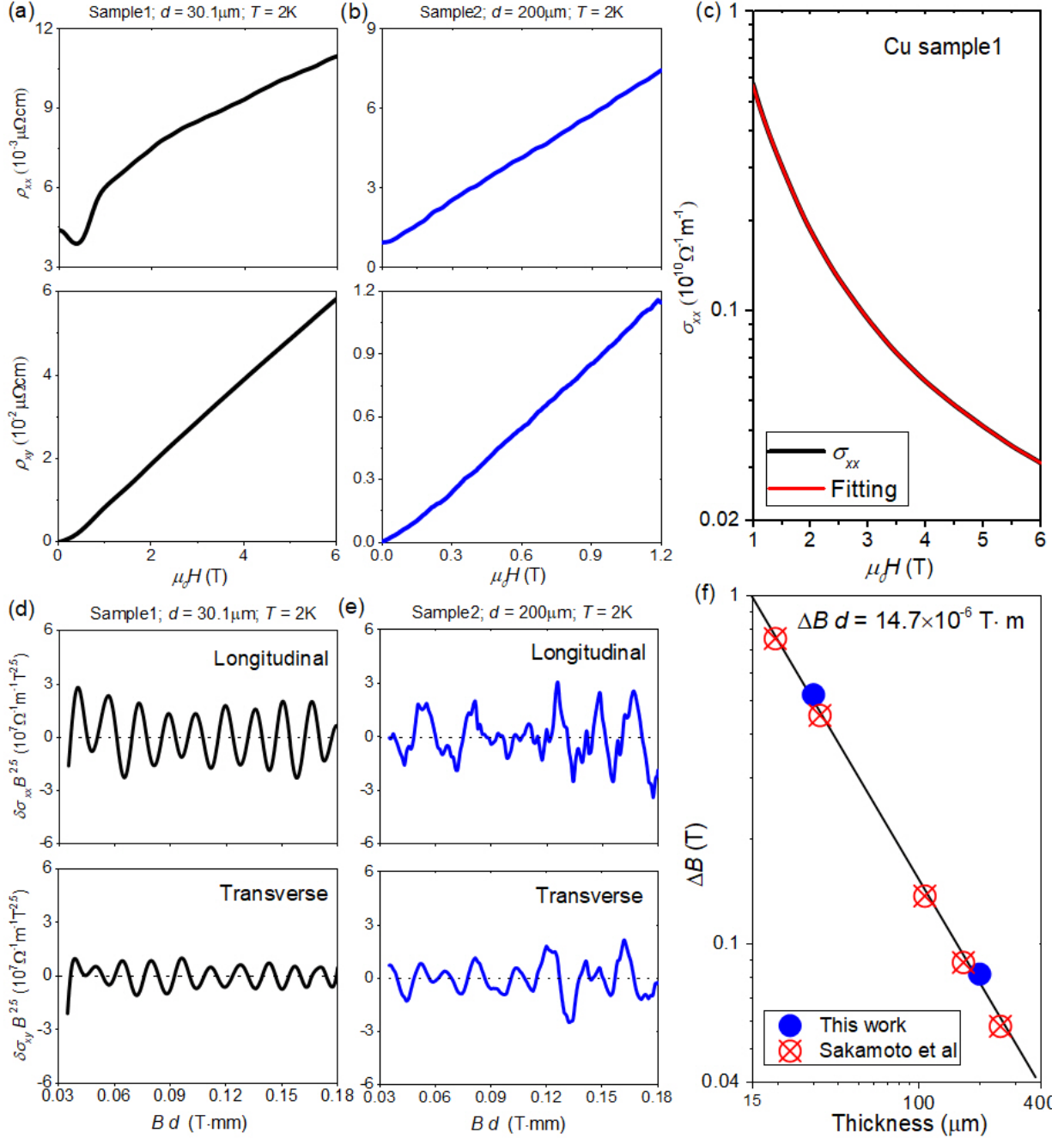


FIG. S6. **Sondheimer oscillations in copper crystals.** (a) Longitudinal (top) and transverse (bottom) resistivity in the $30.1 \mu\text{m}$ sample. Sondheimer oscillations can be observed in the longitudinal resistivity. (b) Same for the $200 \mu\text{m}$ sample. (c) Longitudinal conductivity in $30.1 \mu\text{m}$ sample. The red line represents a polynomial fit to the background. (d) Oscillating component of conductivity multiplied by $B^{2.5}$ as function of the product of magnetic field and thickness (Bd) in a copper crystal with a thickness of $30.1 \mu\text{m}$. (e) Oscillating component of conductivity multiplied by $B^{2.5}$ as function of Bd in another copper crystal with a thickness of $200 \mu\text{m}$. All data were taken at $T = 2\text{K}$ with the field applied along the $[100]$ orientation. (f) The variation of the period of oscillation with thickness in our two samples together with those reported in a previous report [4].

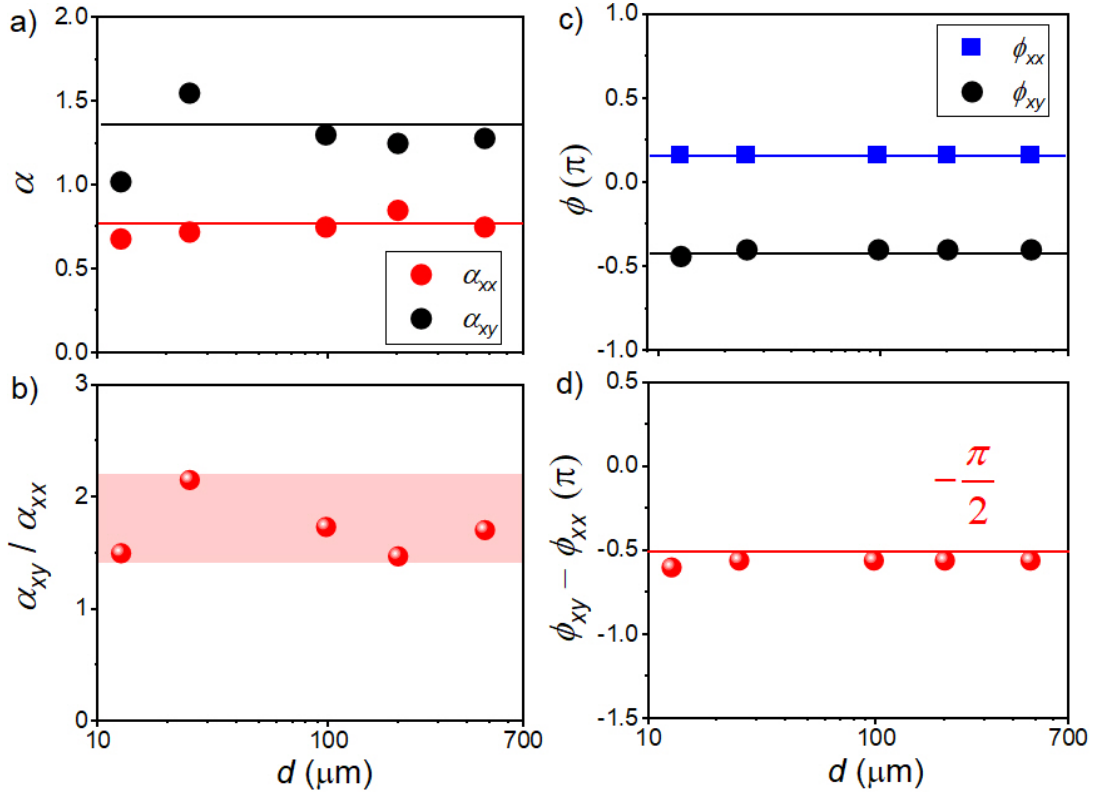


FIG. S7. **Amplitude and phase of oscillations.** Fitting our data to $\alpha \sin(\Omega - \phi)$ for samples with different thickness (Fig.9 in the main text) yields an α (panel a) and a ϕ (panel c) for each sample. **(b)** Ratio of the longitudinal and transverse α is close to two in all samples. **(d)** The shift between ϕ in longitudinal and transverse conductivity is close to $-\frac{\pi}{2}$.

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