Supplement for the paper

Group Formation under Limited Resource: Narrow Basin of Equality

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1. Basic setup for analysis

The size of the first group $n=n_1$ is found as the maximum value of the payoff function

$$\pi_1(n)=R\frac{n^{\alpha-\beta}}{n^{\alpha}+(N-n)^{\alpha}}$$

for $1\le n\le N$. Once n_1 is fixed, then $n=n_2$ is determined as the maximum value of the function

$$\pi_2(n)=R\frac{n^{\alpha-\beta}}{n_1^{\alpha}+n^{\alpha}+(N-n_1-n)^{\alpha}}$$

for $1\le n\le N-n_1$. In the same manner, $n=n_3$ is the maximum value of the function

$$\pi_3(n)=R\frac{n^{\alpha-\beta}}{n_1^{\alpha}+n_2^{\alpha}+n^{\alpha}+(N-n_1-n_2-n)^{\alpha}}$$

for $1\le n\le N-n_1-n_2$.

The above observation can be generalized; once the first $i-1$ groups are determined as $n_1,\cdots,n_{i-1}$, the size of the i-th group $n=n_i$ is found as a maximum of the payoff function

$$\pi_i(n)=R\frac{n^{\alpha-\beta}}{\widetilde{m}+n^{\alpha}+(N-m-n)^{\alpha}}$$

where $m=n_1+\cdots+n_{i-1}$, $\widetilde{m}=n_1^{\alpha}+\cdots+n_{i-1}^{\alpha}$ and $1\le n\le N-m$. Especially if $i=1$, then $m=\widetilde{m}=0$.

For convenience of computation, we rewrite the function

$$\pi_i(r)=\frac{R}{N^{\beta}}\frac{r^{\alpha-\beta}}{\widetilde{r_c}+r^{\alpha}+(1-r_c-r)^{\alpha}},\quad 0\le r\le 1-r_c \tag{1}$$

where $r=n/N$, $r_c=m/N$ and $\widetilde{r_c}=\widetilde{m}/N^{\alpha}$. Now the derivative is obtained as

$$\pi'_i(r)=\frac{R}{N^{\beta}}\frac{r^{\alpha-\beta-1}}{\left(\widetilde{r_c}+r^{\alpha}+\left(1-r_c-r\right)^{\alpha}\right)^2}\,f(r) \tag{2}$$

where

$$f(r)=(\alpha-\beta)\widetilde{r_c}-\beta r^{\alpha}+(\alpha-\beta)(1-r_c-r)^{\alpha}+\alpha(1-r_c-r)^{\alpha-1} \tag{3}$$

for $0\le r\le 1-r_c$.

Note that $\pi_i'(r)$ and $f(r)$ share the same signs.

2. Group distribution with $\alpha\to 1^{+}$ and $\beta=1$.

With $\beta=1$, the equation (3) becomes

$$f(r)=(\alpha-1)\widetilde{r_c}-r^{\alpha}+(\alpha-1)(1-r_c-r)^{\alpha}+\alpha(1-r_c-r)^{\alpha-1}.$$

Note, at the boundary $r=0$, the function f is positive since

$$f(0)=(\alpha-1)\widetilde{r_c}+(\alpha-1)(1-r_c)^{\alpha}+\alpha(1-r_c)^{\alpha-1}>0.$$

At the other boundary $r=1-r_c$, we get

$$f(1-r_c)=(\alpha-1)\widetilde{r_c}-(1-r_c)^{\alpha}.$$

However, since we trace the case $\alpha\to1^+$, the coefficient $(\alpha-1)$ can be treated as an arbitrarily small positive value. This implies $f(1-r_c)<0$. Also note that $f(r)$ is strictly decreasing for $0\le r\le 1-r_c$, as

$$\begin{aligned}f'(r)=&-\alpha r^{\alpha-1}-\alpha(\alpha-1)(1-r_c-r)^{\alpha-1}-\alpha(\alpha-1)(1-r_c-r)^{\alpha-2}\\ &<0.\end{aligned}$$

This implies that $f(r)$ crosses 0 exactly once.

Since the sign and zeros of $\pi_i'(r)$ coincide with those of $f(r)$, we have $\pi_i'(0)>0$, $\pi_i'(1-r_c)<0$ and the strict decrease as well. Hence there exists a value of $0<r<1-r_c$ at which $\pi_i(r)$ has the global maximum with $\pi_i'(r)=0$. This condition is equivalent to $f(r)=0$, or

$$\frac{(\alpha-1)\left(\widetilde{r_c}+(1-r_c-r)^{\alpha}\right)}{r^{\alpha}-\alpha(1-r_c-r)^{\alpha-1}}=1.$$

As $\alpha\to1^+$, from the L'hopital's rule, we get

$$\begin{aligned}1&=\lim_{\alpha\to1^+}\frac{(\alpha-1)\left(\widetilde{r_c}+(1-r_c-r)^{\alpha}\right)}{r^{\alpha}-\alpha(1-r_c-r)^{\alpha-1}}\\ &=\lim_{\alpha\to1^+}\frac{\frac{d}{d\alpha}(\alpha-1)\left(\widetilde{r_c}+(1-r_c-r)^{\alpha}\right)}{\frac{d}{d\alpha}\left(r^{\alpha}-\alpha(1-r_c-r)^{\alpha-1}\right)}\end{aligned}$$

Here r should be treated as a function of α. That is, we need to apply the rule

$$\begin{aligned}\frac{d}{d\alpha}r^{\alpha}&=\left(\frac{dr}{d\alpha}\frac{\partial}{\partial r}+\frac{\partial}{\partial\alpha}\right)r^{\alpha}\\ &=\alpha r^{\alpha-1}r'+r^{\alpha}\ln r.\end{aligned}$$

Noting that $\widetilde{r_c}-r_c^{\alpha}$ converges to 0 as $\alpha\to1$, we get

$$\frac{1-r}{r\ln r-r-r\ln(1-r_c-r)}=1$$

or

$$\frac{1}{r} = \ln \frac{r}{1 - r_c - r} \tag{4}$$

The equation (4) uniquely determines the value of r from r_c. (The uniqueness of r guarantees that such r is the global maximum point)

Applying (4) as an implicit recurrence relation from $r_c = \dfrac{n_1 + \cdots + n_{i-1}}{N}$ to $r = \dfrac{n_i}{N}$ repeatedly gives $(0.7857, 0.2157, 0.0021, \cdots)$. It is clear that the corresponding individual payoff for each group is identical since

$$\pi_i(r) \to \frac{R}{N} \quad \text{as } \alpha \to 1$$

regardless of r_c.

3. Numerical examples for $\alpha = 1.4, 1.5, 1.6$ with $N = 10$ and $\beta = 1.5$.

i) $\alpha = 1.4$ and $\beta = 1.5$

The payoff function for the members of the first group is

$$\pi_1(n) = R \frac{n^{-0.1}}{n^{1.4} + (10-n)^{1.4}}, \quad n = 1, 2, \cdots, 10.$$

One can numerically confirm that this has the maximum at $n = 4$. Then π_2 for the second group is

$$\pi_2(n) = R \frac{n^{-0.1}}{4^{1.4} + n^{1.4} + (6-n)^{1.4}}, \quad n = 1, 2, \cdots, 6,$$

which attains the maximum at $n = 1$. Then, again,

$$\pi_3(n) = R \frac{n^{-0.1}}{4^{1.4} + 1^{1.4} + n^{1.4} + (5-n)^{1.4}}, \quad n = 1, 2, \cdots, 5.$$

This has also the maximum at $n = 1$. One can further confirm that the same situation (the maximum at $n = 1$) repeats until π_7.

ii) $\alpha = 1.5$ and $\beta = 1.5$

$$\pi_1(n) = R \frac{1}{n^{1.5} + (10-n)^{1.5}}, \quad n = 1, 2, \cdots, 10.$$

This has the maximum at $n = 5$. Then π_2 for the second group is

$$\pi_2(n) = R \frac{1}{5^{1.5} + n^{1.5} + (5-n)^{1.5}}, \quad n = 1, 2, \cdots, 5,$$

which attains the maximum at $n = 2$.

$$\pi_3(n) = R \frac{1}{5^{1.5} + 2^{1.5} + n^{1.5} + (3-n)^{1.4}}, \quad n = 1, 2, 3.$$

This has the maximum at $n = 1$. One can further confirm that the same situation (the maximum at $n=1$) repeats until π_5.

iii) $\alpha=1.6$ and $\beta=1.5$

$$\pi_1(n)=R\frac{n^{0.1}}{n^{1.6}+(10-n)^{1.6}}, \quad n=1,2,\cdots,10.$$

This has the maximum at $n=5$. Then π_2 for the second group is

$$\pi_2(n)=R\frac{n^{0.1}}{5^{1.6}+n^{1.6}+(5-n)^{1.6}}, \quad n=1,2,\cdots,5,$$

which attains the maximum at $n=3$.

$$\pi_3(n)=R\frac{n^{0.1}}{5^{1.6}+3^{1.6}+n^{1.6}+(2-n)^{1.4}}, \quad n=1,2.$$

This has the maximum at $n=2$.

4. Proof of Theorem 1.

(a) $\alpha<\beta/2$

Note that, if $f(r)<0$ for $0\le r\le 1-r_c$, then $\pi_i(r)$ has a maximum at its low boundary, which is $1/N$.

To show $f(r)<0$ for $0\le r\le 1-r_c$, define

$$g(y)=-\beta+(\alpha-\beta)y^{\alpha}+\alpha y^{\alpha-1}$$

where $y=\dfrac{1-r_c-r}{r}$.

Then

$$f(r)=(\alpha-\beta)\widetilde{r_c}+r^{\alpha}g(y)$$

Since $2\alpha-\beta<0$, it suffices to show $g(y)<0$ for $y\ge 0$.

This is clear for $0\le y\le 1$. If $y\ge 1$,

$$\begin{aligned} g(y) &= ((\alpha-\beta)y+\alpha)y^{\alpha-1}-\beta \\ &< (2\alpha-\beta)y^{\alpha-1}-\beta \\ &< 0 \end{aligned}$$

(b) $\alpha=\beta>1$

Now the equation (1) becomes

$$\pi_i(r)=\frac{R}{N}\frac{1}{\widetilde{r_c}+r^{\alpha}+(1-r_c-r)^{\alpha}}$$

From the symmetry over $0\le r\le 1-r_c$, the payoff has the maximum at

$r = \dfrac{1-r_c}{2}$. This implies that the group size is always the half of the remaining population which is unorganized yet.

(c) $\alpha \to \infty$

Note $r_c = 0$ and $\widetilde{r_c} = 0$ for the first group. The payoff of the first group $\pi_1(r)$ has a critical point at $r = r_*$ such that

$$f(r_*) = -\beta r_*^{\alpha} + (\alpha - \beta)(1-r_*)^{\alpha} + \alpha(1-r_*)^{\alpha-1} = 0.$$

For sufficiently large α, this r_* is indeed the global maximum, as $f(0) = 2\alpha - \beta > 0$, $f(1) = -\beta < 0$, and

$$f'(r) = -\alpha\beta r^{\alpha-1} - \alpha(\alpha-\beta)(1-r)^{\alpha-1} - \alpha(\alpha-1)(1-r)^{\alpha-2}$$

$$< 0$$

for $0 \le r \le 1$.

Now, for such r_*, we observe

$$0 = f(r_*)/r_*^{\alpha} \qquad (5)$$

$$= -\beta + (\alpha-\beta)y^{\alpha} + \alpha y^{\alpha-1} = 0$$

where $y = \dfrac{1-r_*}{r_*}$, $y \ge 0$.

It is clear from (5) that y converges to 1^- or less as $\alpha \to \infty$. This implies that, as $\alpha \to \infty$, r_* converges to $\left(\dfrac{1}{2}\right)^{+}$ or greater, or equivalently, $n_1 > N/2$ for sufficiently large α.

Now let us prove the following statement:

For any $n > N/2$,

$$\pi_1(n) > \pi_1(n+1) \text{ for sufficiently large } \alpha. \qquad (6)$$

This statement, together with the above result, leads to the conclusion that $n_1 = k+1$ in the limit where $N = 2k$ or $2k+1$.

One can observe the inequality (6) is equivalent to

$$\frac{n^{\alpha}}{(n+1)^{\alpha}} \frac{(n+1)^{\alpha} + (N-n-1)^{\alpha}}{n^{\alpha} + (N-n)^{\alpha}} > \frac{n^{\beta}}{(n+1)^{\beta}}$$

or

$$\frac{1+(N/(n+1)-1)^{\alpha}}{1+(N/n-1)^{\alpha}} > \frac{n^{\beta}}{(n+1)^{\beta}}. \tag{7}$$

The right hand side is a constant strictly less than 1. The left hand side is also less than 1. However, since $N/n-1<1$, the left hand side converges to 1 from the below as $\alpha \rightarrow \infty$. This implies that one can make the left hand side arbitrarily close to 1. Hence the equation (7) (and therefore (6)) is true for sufficiently large α.

Now let us show that the second group takes all the remaining population. Suppose the total population is $N=2k+1$(The proof for the case for $N=2k$ is similar.) We assume that α is large enough so that $n_1=k+1$ as above. Then the payoff function for the second group is

$$\pi_2(n)=R\frac{n^{\alpha-\beta}}{(k+1)^{\alpha}+n^{\alpha}+(k-n)^{\alpha}}$$

where $1 \le n \le k$.

Now let us show that the maximum of $\pi_2(n)$ occurs at $n=k$ by proving the following statement:

For any $1 \le n \le k-1$,

$$\pi_2(n) < \pi_2(n+1) \text{ for sufficiently large } \alpha. \tag{8}$$

One can confirm that the inequality (8) is equivalent to

$$\frac{n^{\alpha}}{(n+1)^{\alpha}}\frac{(k+1)^{\alpha}+(n+1)^{\alpha}+(k-n-1)^{\alpha}}{(k+1)^{\alpha}+n^{\alpha}+(k-n)^{\alpha}} < \frac{n^{\beta}}{(n+1)^{\beta}}$$

or

$$\frac{n^{\alpha}}{(n+1)^{\alpha}}\frac{1+((n+1)/(k+1))^{\alpha}+((k-n-1)/(k+1))^{\alpha}}{1+(n/(k+1))^{\alpha}+((k-n)/(k+1))^{\alpha}} < \frac{n^{\beta}}{(n+1)^{\beta}} \tag{9}$$

The left hand side of (9) converges to 0 as $\alpha \rightarrow \infty$. Hence, the inequality of (8) holds for sufficiently large α.

5. Proof of Theorem 2.

(a) All groups are identical as $n_1=n_2=\cdots=1$ and therefore they share the same group gain,

$$g_i=\pi_i=\frac{R}{N},$$

which results in the unit indices.

(b) From the group distribution $\overline{D}=(\frac{1}{2},\frac{1}{4},\cdots)$, the normalized group gain of the ith group is

$$\overline{g_i}=\frac{(1/2)^{\alpha i}}{(1/2)^{\alpha}+(1/2)^{2\alpha}+\cdots}=(1/2)^{\alpha(i-1)}(1-(1/2)^{\alpha}).$$

One can see

$$\frac{\overline{g_i}}{\overline{n_i}}=(1/2)^{\alpha(i-1)-\alpha}(1-(1/2)^{\alpha})$$

has the maximum $2-2^{1-\alpha}$ at $i=1$, which gives r_P.

The individual payoff of i-th group is evaluated as

$$\pi_i=R\frac{(N/2)^{(\alpha-\beta)i}}{(N/2)^{\alpha}+(N/4)^{\alpha}+\cdots}=\frac{R}{N^{\alpha}}(2^{\alpha}-1).$$

However, the result is a constant and do not change with a group. Hence $r_R=1$.

(c)

In the proof of Theorem 1(c), we confirmed that $n_1\to\left(\frac{N}{2}\right)^{+}$ and $n_2\to\left(\frac{N}{2}\right)^{-}$ as $\alpha\to\infty$. This leads to $\frac{n_2}{n_1}=\frac{\overline{n_2}}{\overline{n_1}}\to 1^{-}$

and moreover

$$\begin{aligned}\overline{g}_1&=\frac{n_1^{\alpha}}{n_1^{\alpha}+n_2^{\alpha}}\\&=\frac{1}{1+\left(\frac{n_2}{n_1}\right)^{\alpha}}\\&\to 1\end{aligned}$$

as $\alpha\to\infty$. This means that the first group takes the all resource while the other one takes nothing in the limit. Hence we have

$$r_P=\frac{\overline{g_1}}{\overline{n_1}}=\frac{1}{\frac{1}{2}}=2$$

and

$$r_R=\frac{\pi_1}{\pi_2}=\frac{2}{0}=\infty.$$