Appendix A: Time derivatives of the thermodynamic variables

In addition to discussing the choice of hydrodynamic frame to accurately define thermodynamic variables, the description of fluid motion necessitates the selection of a specific set of thermodynamic variables to be incorporated into the equations of motion. For a fluid with a single type of charge, there are four main thermodynamic quantities: energy density ϵ, number density n, pressure p, and entropy density s. The zeroth order equations of motion provide relations for the gradients of energy density and number density in terms of fluid scalar expansion Θ. It is desirable to employ the equation of state for pressure, denoted $p = p(T, \mu)$, to express these equations in terms of temperature and chemical potential gradients. Given that $s = (\partial p/\partial T)_\mu$, $n = (\partial p/\partial \mu)_T$, and $\epsilon = -p + Ts + \mu n$, we can express s, n, and ϵ as functions of T and μ. These can be combined with Eq. (4) and the first equation in the set of Eqs. (5) to yield the following relations:

$$\left(\frac{\partial \epsilon}{\partial T}\right)_\mu DT + \left(\frac{\partial \epsilon}{\partial \mu}\right)_T D\mu = -h\Theta, \tag{A.1}$$

$$\left(\frac{\partial n}{\partial T}\right)_\mu DT + \left(\frac{\partial n}{\partial \mu}\right)_T D\mu = -n\Theta. \tag{A.2}$$

The linear structure of this set of equations indicates that a particular criterion must be met for the equations to be used to eliminate the time derivatives of μ and T in terms of $\Theta = \nabla_{\perp\alpha} u^\alpha$. To clarify, we express the foregoing equations in matrix form:

$$\begin{pmatrix} \epsilon_{,T} & \epsilon_{,\mu} \\ n_{,T} & n_{,\mu} \end{pmatrix} D \begin{pmatrix} T \\ \mu \end{pmatrix} = - \begin{pmatrix} h \\ n \end{pmatrix} \Theta, \tag{A.3}$$

where the operator D act on each element of the matrix to the right and we have used a simplified notation for partial derivatives, $\epsilon_{,T} = (\partial\epsilon/\partial T)_\mu$, and so on. This system of equations is solvable if and only if the 2×2 matrix on the left-hand side is invertible. Equivalently, this condition implies that the determinant of the matrix must be nonzero:

$$\epsilon_{,T} n_{,\mu} - \epsilon_{,\mu} n_{,T} \neq 0. \tag{A.4}$$

Once this condition is satisfied, one can use the equations of motion of an ideal fluid to eliminate the time derivatives of the Lorentz scalars T, μ.

In the case of a multiphase fluid with N particle species, we encounter $N+1$ variables $\epsilon, n_1, n_2, ..., n_N$ that are functions of the $N+1$ scalars $T, \mu_1, \mu_2, ..., \mu_N$. For each species, there exists a corresponding continuity equation, resulting in $N+1$ equations that can be represented in matrix form, generalizing Eq. (A.3):

$$\begin{pmatrix} \epsilon_{,T} & \epsilon_{,\mu_1} & \cdots & \epsilon_{,\mu_N} \\ n_{1,T} & n_{1,\mu_1} & \cdots & n_{1,\mu_N} \\ \vdots & \vdots & \ddots & \vdots \\ n_{N,T} & n_{N,\mu_1} & \cdots & n_{N,\mu_N} \end{pmatrix} D \begin{pmatrix} T \\ \mu_1 \\ \vdots \\ \mu_N \end{pmatrix} = - \begin{pmatrix} h \\ n_1 \\ \vdots \\ n_N \end{pmatrix} \Theta. \tag{A.5}$$

In this scenario, the necessary and sufficient condition for expressing time derivatives in terms of the scalar expansion is that the matrix on the left-hand side of the above equation be invertible, or equivalently, its determinant must be nonzero.

Appendix B: Equivalences between symmetric gradients

It is well established that a rank-two tensor can be Weyl-decomposed into three components, each defined by its symmetry properties under the rotation group: the antisymmetric part, the symmetric and traceless part, and the scalar (or trace) part. In the realm of hydrodynamics, it transpires that a fundamental gradient is represented as a rank-two tensor, which we then subject to Weyl decomposition:

$$\nabla^\alpha u^\beta = \sigma^{\alpha\beta} + \Omega^{\alpha\beta} + \frac{1}{d-1}\Theta\Delta^{\alpha\beta}. \tag{B.6}$$

Each term in this decomposition has a distinct physical meaning. The symmetric and traceless tensor (TST) part, denoted by σ, encodes shear effects; the trace part, Θ, accounts for expansion and compression; and the antisymmetric part, Ω, represents vorticity. The physical content of these terms provides a strong motivation to use σ, Θ, and Ω as fundamental structures in the building of the gradient expansion. However, a nontrivial issue arises. It has been demonstrated in [42] that the ordering of transverse derivatives is irrelevant in the construction of the tensorial structures of the hydrodynamic expansion. If one opts not to decompose the velocity gradient, this equivalence is easily implemented by establishing $\nabla^\alpha\nabla^\beta u^\gamma \simeq \nabla^\beta\nabla^\alpha u^\gamma$. On the contrary, translating this rule of equivalence to the gradients of σ, Θ, and Ω presents a challenge.

It should be noted that the second derivative of the velocity field can be expressed as a linear combination of the gradients of σ, Θ, and Ω. Together with the irrelevance of derivative ordering mentioned above, this property suggests an underlying relationship among the gradients of these tensor components. In fact, using the notation of [42], one can formally establish that

$$\nabla_\perp^\alpha \Omega^{\beta\gamma} \simeq \nabla_\perp^\beta \left(\sigma^{\alpha\gamma} + \frac{\Delta^{\alpha\gamma}}{d-1}\Theta\right) - \nabla_\perp^\gamma \left(\sigma^{\alpha\beta} + \frac{\Delta^{\alpha\beta}}{d-1}\Theta\right). \tag{B.7}$$

In the above equation, the symbol $\simeq$ indicates that the tensors on both sides of the equation are equivalent, meaning their difference constitutes a linear combination of other gradients already accounted for in the gradient expansion. For example, the divergence of vorticity satisfies $\nabla_{\perp\alpha}\Omega^{\alpha\gamma} \simeq \nabla_{\perp\alpha}\sigma^{\alpha\gamma} + 2\nabla_\perp^\gamma\Theta/(d-1)$. Practically speaking, the implication of Eq. (B.7) is that one may elect to omit vorticity gradients from the terms of the hydrodynamic expansion, substituting them by the shear and expansion gradients. Such an omission of vorticity derivatives is also coherent in the context of conformal fluids, where the expansion is identically null.

Appendix C: General-frame derivation of dispersion relations

In this appendix, we provide an alternative derivation of the dispersion relations expressed in (130), (131) and (132). This derivation relies on constitutive relations written in a general hydrodynamic frame. More specifically, we consider the decomposition of the dissipative terms $\Pi^{\alpha\beta}$ and Υ^{α} as follows:

$$\Pi^{\alpha\beta} = \mathcal{E}u^{\alpha}u^{\beta} + \mathcal{P}\Delta^{\alpha\beta} + 2\mathcal{Q}^{(\alpha}u^{\beta)} + \tau^{\alpha\beta}, \tag{C.8}$$
$$\Upsilon^{\alpha} = \mathcal{N}u^{\alpha} + \mathcal{J}^{\alpha}. \tag{C.9}$$

Each term in these equations is expanded up to the third order in the gradient series,

$$\begin{aligned}
\mathcal{E} &= \mathcal{E}_{(1)} + \mathcal{E}_{(2)} + \mathcal{E}_{(3)},\\
\mathcal{P} &= \mathcal{P}_{(1)} + \mathcal{P}_{(2)} + \mathcal{P}_{(3)},\\
\mathcal{N} &= \mathcal{N}_{(1)} + \mathcal{N}_{(2)} + \mathcal{N}_{(3)},\\
\mathcal{Q}^{\alpha} &= \mathcal{Q}^{\alpha}_{(1)} + \mathcal{Q}^{\alpha}_{(2)} + \mathcal{Q}^{\alpha}_{(3)},\\
\mathcal{J}^{\alpha} &= \mathcal{J}^{\alpha}_{(1)} + \mathcal{J}^{\alpha}_{(2)} + \mathcal{J}^{\alpha}_{(3)},\\
\tau^{\alpha\beta} &= \tau^{\alpha\beta}_{(1)} + \tau^{\alpha\beta}_{(2)} + \tau^{\alpha\beta}_{(3)}.
\end{aligned} \tag{C.10}$$

Owing to the computational complexity and the extensive nature of the calculations in a general frame, this appendix focuses on expanding the aforementioned coefficients in terms of gradients of the densities ϵ and n, as opposed to temperature and chemical potential. Only after deriving the dispersion relations do we substitute the relationships between the transport coefficients associated with the pair (ϵ, n) and those corresponding to (T, μ).

The equations of motion comprise one vector equation and two scalar equations. By taking the vanishing divergence of $T^{\alpha\beta}$ and projecting it in directions parallel and perpendicular to u^{α}, we derive the scalar equation (98) and the vector equation (99). The remaining scalar equation, Eq. (100), emerges from the vanishing divergence of J^{α}. Substituting (C.8) and (C.9) into Eqs. (98), (99), and (100), we arrive at the following equations:

$$\begin{aligned}
&D(\epsilon + \mathcal{E}) + (h + \mathcal{E} + \mathcal{P})\Theta\\
&\qquad + \tau^{\alpha\beta}\sigma_{\alpha\beta} + \nabla_{\alpha}\mathcal{Q}^{\alpha} + \mathcal{Q}_{\alpha}Du^{\alpha} = 0,
\end{aligned} \tag{C.11}$$
$$\begin{aligned}
&(h + \mathcal{E} + \mathcal{P})Du^{\alpha} + \nabla^{\alpha}_{\perp}(p + \mathcal{P}) + \Delta^{\alpha}{}_{\beta}\nabla_{\gamma}\tau^{\gamma\beta}\\
&\qquad + \mathcal{Q}^{\beta}\nabla_{\beta}u^{\alpha} + \Delta^{\alpha}{}_{\beta}D\mathcal{Q}^{\beta} + \mathcal{Q}^{\alpha}\Theta = 0,
\end{aligned} \tag{C.12}$$
$$(n + \mathcal{N})\Theta + D(n + \mathcal{N}) + \nabla_{\alpha}\mathcal{J}^{\alpha} = 0. \tag{C.13}$$

By retaining only the linear terms in the amplitudes, the dissipation structures in the above equations can be expressed in terms of transport coefficients as follows:

$$\begin{aligned}
\mathcal{E} &= \left(\varepsilon^{(1)}_{\Theta} + \varepsilon^{(3)}_{\Theta}\nabla^2_{\perp}\right)\Theta + \sum_a \varepsilon^{(2)}_a \nabla^2_{\perp} a,\\
\mathcal{P} &= \left(\pi^{(1)}_{\Theta} + \pi^{(3)}_{\Theta}\nabla^2_{\perp}\right)\Theta + \sum_a \pi^{(2)}_a \nabla^2_{\perp} a,\\
\mathcal{N} &= \left(\nu^{(1)}_{\Theta} + \nu^{(3)}_{\Theta}\nabla^2_{\perp}\right)\Theta + \sum_a \nu^{(2)}_a \nabla^2_{\perp} a,\\
\mathcal{Q}^{\alpha} &= \sum_a \left(\theta^{(1)}_a + \theta^{(3)}_a\nabla^2_{\perp}\right)\nabla^{\alpha}_{\perp} a\\
&\quad + \theta^{(2)}_{\Theta}\nabla^{\alpha}_{\perp}\Theta + \theta^{(2)}_{\sigma}\Delta^{\alpha}{}_{\beta}\nabla_{\perp\gamma}\sigma^{\gamma\beta},\\
\mathcal{J}^{\alpha} &= \sum_a \left(\kappa^{(1)}_a + \kappa^{(3)}_a\nabla^2_{\perp}\right)\nabla^{\alpha}_{\perp} a\\
&\quad + \kappa^{(2)}_{\Theta}\nabla^{\alpha}_{\perp}\Theta + \kappa^{(2)}_{\sigma}\Delta^{\alpha}{}_{\beta}\nabla_{\perp\gamma}\sigma^{\gamma\beta},\\
\tau^{\alpha\beta} &= \left(\eta^{(1)}_{\sigma} + \eta^{(3)}_{\sigma}\nabla^2_{\perp}\right)\sigma^{\alpha\beta}\\
&\quad + \sum_a \eta^{(2)}_a \nabla^{\langle\alpha}_{\perp}\nabla^{\beta\rangle}_{\perp} a + \eta^{(3)}_{\Theta}\nabla^{\langle\alpha}_{\perp}\nabla^{\beta\rangle}_{\perp}\Theta,
\end{aligned} \tag{C.14}$$

where a takes the value ϵ and n. As a mnemonic strategy, we choose to use the subscripts Θ, σ, ϵ, and n throughout this appendix, rather than the numerical indices used to denote various transport coefficients in Sections 2 and 3.

In the case of a d-dimensional flat spacetime and considering the linear fluctuations in the hydrodynamic variables defined in (106), Eqs. (C.11)–(C.13) reduce to

$$u^{\alpha}\partial_{\alpha}(\delta\epsilon + \delta\mathcal{E}) + h\partial_{\perp\alpha}\delta u^{\alpha} + \partial_{\perp\alpha}\delta\mathcal{Q}^{\alpha} = 0, \tag{C.15}$$
$$hu^{\beta}\partial_{\beta}\delta u^{\alpha} + u^{\beta}\partial_{\beta}\delta\mathcal{Q}^{\alpha} + \partial^{\alpha}_{\perp}(\delta p + \delta\mathcal{P}) + \partial_{\perp\beta}\delta\tau^{\beta\alpha} = 0, \tag{C.16}$$
$$u^{\alpha}\partial_{\alpha}(\delta n + \delta\mathcal{N}) + n\partial_{\perp\alpha}\delta u^{\alpha} + \partial_{\perp\alpha}\delta\mathcal{J}^{\alpha} = 0. \tag{C.17}$$

By Fourier-decomposing the perturbations in Eqs. (C.15)–(C.17) in terms of the plane waves $\exp(ik_{\alpha}x^{\alpha})$, we obtain

$$k_{\parallel}(\delta\epsilon + \delta\mathcal{E}) + k_{\perp\alpha}(h\delta u^{\alpha} + \delta\mathcal{Q}^{\alpha}) = 0, \tag{C.18}$$
$$k_{\parallel}(h\delta u^{\alpha} + \delta\mathcal{Q}^{\alpha}) + k^{\alpha}_{\perp}(\delta p + \delta\mathcal{P}) + k_{\perp\beta}\delta\tau^{\beta\alpha} = 0, \tag{C.19}$$
$$k_{\parallel}(\delta n + \delta\mathcal{N}) + k_{\perp\alpha}(n\delta u^{\alpha} + \delta\mathcal{J}^{\alpha}) = 0, \tag{C.20}$$

where $k_{\parallel} = u^{\alpha}k_{\alpha}$ and $k^{\alpha}_{\perp} = \Delta^{\alpha\beta}k_{\beta}$. The perturbations in $\mathcal{E}$, $\mathcal{P}, \mathcal{N}, \mathcal{Q}^{\alpha}, \mathcal{J}^{\alpha}$ and $\tau^{\alpha\beta}$ are given by the Fourier-transformed linearized versions of equations (C.14). Furthermore, the perturbation in the pressure assumes the form:

$$\delta p = \beta_{\epsilon}\delta\epsilon + \beta_n\delta n. \tag{C.21}$$

Equations (C.18)–(C.20) constitute a set of two scalar equations and one vector equation for the variables $\delta\epsilon$, δn, and δu^{α}. Projecting the vector equation (C.19) along the direction transverse to $k^{\alpha}_{\perp}$ yields

$$\left[k_{\parallel}\left(2h - \theta^{(2)}_{\sigma}k^2_{\perp}\right) + ik^2_{\perp}(\eta^{(1)}_{\sigma} - \eta^{(3)}_{\sigma}k^2_{\perp})\right]\delta u^{\alpha}_{T} = 0, \tag{C.22}$$

where δu_T^α is defined as

$$\delta u_T^\alpha = \left(\Delta^\alpha{}_\beta - \frac{k_\perp^\alpha k_{\perp\beta}}{k_\perp^2}\right)\delta u^\beta. \tag{C.23}$$

Employing a similar procedure, but projecting (C.19) along the direction longitudinal to $k_\perp^\alpha$, leads to the following equation:

$$\begin{aligned}&\left[\left(\xi_s^{(3)} k_\perp^2 + \lambda_s^{(1)}\right) k_\perp^2 + i\left(\theta_s^{(2)} k_\perp^2 - 1\right) k_\parallel\right] h\delta u_L \\ &+ \sum_a \left[i\left(\gamma_a^{(2)} h k_\perp^2 + \beta_a\right) - \left(\theta_a^{(3)} k_\perp^2 - \theta_a^{(1)}\right) k_\parallel\right] k_\perp \delta a = 0,\end{aligned} \tag{C.24}$$

where the index a takes the values ϵ or n. The set of combinations of transport coefficients introduced above are given by

$$\begin{aligned}\lambda_s^{(1)} &= -\frac{1}{h}\left(\pi_\Theta^{(1)} + \frac{d-2}{d-1}\eta_\sigma^{(1)}\right),\\ \gamma_a^{(2)} &= -\frac{1}{h}\left(\pi_a^{(2)} + \frac{d-2}{d-1}\eta_a^{(2)}\right),\\ \theta_s^{(2)} &= \frac{1}{h}\left(\theta_\Theta^{(2)} + \frac{d-2}{d-1}\theta_\sigma^{(2)}\right),\\ \xi_s^{(3)} &= \frac{1}{h}\left(\pi_\Theta^{(3)} + \frac{d-2}{d-1}\left(\eta_\Theta^{(3)} + \eta_\sigma^{(3)}\right)\right).\end{aligned} \tag{C.25}$$

Additionally, δu_L denotes the magnitude of the longitudinal projection vector of the fluid velocity perturbation, defined as

$$\delta u_L^\alpha = \frac{(k_{\perp\beta}\delta u^\beta)}{k_\perp^2} k_\perp^\alpha. \tag{C.26}$$

Note that the frame-invariant quantities $\gamma_s^{(1)}$ and $\chi_s^{(3)}$, defined in (120), are equal, respectively, to $\lambda_s^{(1)}$ and $\xi_s^{(3)}$ in the Landau frame.

In terms of δu_L, $\delta\epsilon$, and δn, equations (C.18) and (C.20) can be reformulated as

$$\begin{aligned}&\left[ih\left(\theta_s^{(2)} k_\perp^2 - 1\right) + \left(\varepsilon_\Theta^{(3)} k_\perp^2 - \varepsilon_\Theta^{(1)}\right) k_\parallel\right] k_\perp \delta u_L \\ &+ik_\parallel \delta\epsilon - \sum_a \left(\theta_a^{(3)} k_\perp^2 - \theta_a^{(1)} + i\varepsilon_a^{(2)} k_\parallel\right) k_\perp^2 \delta a = 0,\end{aligned} \tag{C.27}$$

$$\begin{aligned}&\left[ih\left(\kappa_s^{(2)} k_\perp^2 - 1\right) + \left(\nu_\Theta^{(3)} k_\perp^2 - \nu_\Theta^{(1)}\right) k_\parallel\right] k_\perp \delta u_L \\ &+ik_\parallel \delta n - \sum_a \left(\kappa_a^{(3)} k_\perp^2 - \kappa_a^{(1)} + i\nu_a^{(2)} k_\parallel\right) k_\perp^2 \delta a = 0,\end{aligned} \tag{C.28}$$

where $\kappa_s^{(2)}$ is defined as

$$\kappa_s^{(2)} = \frac{1}{h}\left(\kappa_\Theta^{(2)} + \frac{d-2}{d-1}\kappa_\sigma^{(2)}\right). \tag{C.29}$$

Together with (C.24), equations (C.27) and (C.28) comprise a system of three homogeneous equations for the independent variables δu_L, $\delta\epsilon$, and δn. From the vanishing of the determinant of the matrix of coefficients of this system of equations, as well as of the term within square brackets in (C.22), we respectively derive the equations for the longitudinal and transverse hydrodynamic modes.

To find the dispersion relations, we adopt the rest reference frame of the fluid,

$$u^\alpha = (1, \vec{0}) \;\Rightarrow\; k_\parallel = u^0 k_0 \equiv -\omega \;\text{ and }\; k_\perp^2 = k^i k_i \equiv k^2. \tag{C.30}$$

In this manner, the algebraic equations for the hydrodynamic modes transform into

$$\begin{aligned}\omega\left(2h - \theta_\sigma^{(2)} k^2\right) + ik^2(\eta_\sigma^{(1)} - \eta_\sigma^{(3)} k^2) &= 0,\\ \det \mathcal{M}(\omega, k) &= 0,\end{aligned} \tag{C.31}$$

where the elements of the matrix $\mathcal{M}$ are defined as

$$\begin{aligned}\mathcal{M}_{11} &= \left[\left(\xi_s^{(3)} k^2 + \lambda_s^{(1)}\right) k^2 + i\left(\theta_s^{(2)} k^2 - 1\right)\omega\right] h,\\ \mathcal{M}_{12} &= \left[i\left(\gamma_\epsilon^{(2)} h k^2 + \beta_\epsilon\right) - \left(\theta_\epsilon^{(3)} k^2 - \theta_\epsilon^{(1)}\right)\omega\right] k,\\ \mathcal{M}_{13} &= \left[i\left(\gamma_n^{(2)} h k^2 + \beta_n\right) - \left(\theta_n^{(3)} k^2 - \theta_n^{(1)}\right)\omega\right] k,\\ \mathcal{M}_{21} &= \left[ih\left(\theta_s^{(2)} k^2 - 1\right) + \left(\varepsilon_\Theta^{(3)} k^2 - \varepsilon_\Theta^{(1)}\right)\omega\right] k,\\ \mathcal{M}_{22} &= -\left(\theta_\epsilon^{(3)} k^2 - \theta_\epsilon^{(1)}\right) k^2 - i\left(\varepsilon_\epsilon^{(2)} k^2 - 1\right)\omega,\\ \mathcal{M}_{23} &= -\left(\theta_n^{(3)} k^2 - \theta_n^{(1)} + i\varepsilon_n^{(2)}\omega\right) k^2,\\ \mathcal{M}_{31} &= \left[i\left(h\kappa_s^{(2)} k^2 - n\right) + \left(\nu_\Theta^{(3)} k^2 - \nu_\Theta^{(1)}\right)\omega\right] k,\\ \mathcal{M}_{32} &= -\left(\kappa_\epsilon^{(3)} k^2 - \kappa_\epsilon^{(1)} + i\nu_\epsilon^{(2)}\omega\right) k^2,\\ \mathcal{M}_{33} &= -\left(\kappa_n^{(3)} k^2 - \kappa_n^{(1)}\right) k^2 - i\left(\nu_n^{(2)} k^2 - 1\right)\omega.\end{aligned} \tag{C.32}$$

The solution to the first of Eqs. (C.31) gives rise to the shear mode associated with the diffusion of transverse momentum in the fluid,

$$\omega_{\text{shear}}(k) = i\frac{\eta_\sigma^{(1)}}{2h}k^2 - \frac{i}{2h}\left(\eta_\sigma^{(3)} - \frac{\eta_\sigma^{(1)}\theta_\sigma^{(2)}}{2h}\right)k^4, \tag{C.33}$$

which reduces to (130), given that $t_1^{(1)} = \eta_\sigma^{(1)}$ and $t_2^{(3)} = \eta_\sigma^{(3)} - \eta_\sigma^{(1)}\theta_\sigma^{(2)}/2h$, in accordance with Eqs. (36) and (72). The solutions to the second of Eqs. (C.31) yield the dispersion relation for the charge diffusion mode, given by

$$\omega_{\text{diffu}}(k) = -i\mathscr{D}k^2 - \frac{i}{v_s^2}\left\{\mathscr{D}\left[v_s^2(\mathscr{A} - \theta_s^{(2)}) + \mathscr{B}\right] + \mathscr{C}\right\}k^4, \tag{C.34}$$

and the dispersion relations for the sound wave modes, which can be expressed as

$$\begin{aligned}\omega_{\text{sound}}^{(\pm)}(k) = \pm v_s k - \frac{i}{2}\Gamma k^2 \mp \frac{1}{8v_s}\left(\Gamma^2 + 4\mathscr{B}\right)k^3 \\ - \frac{i}{2v_s^2}\left[v_s^2\left(\Gamma\mathscr{A} + \mathscr{E}\right) - \mathscr{D}\mathscr{B}\right]k^4,\end{aligned} \tag{C.35}$$

where the first-order diffusion and sound-wave damping constants,

$$\mathscr{D} = \frac{\sigma}{v_s^2}\left(\alpha_n^{(1)}\beta_\epsilon - \alpha_\epsilon^{(2)}\beta_n\right)$$
$$\Gamma = \lambda_s^{(1)} + \frac{\sigma\beta_n}{v_s^2}\left(\alpha_\epsilon^{(1)} + \frac{n}{h}\alpha_n^{(1)}\right) + \vartheta_\Theta^{(1)}, \tag{C.36}$$

correspond to those presented in Section 4, but naturally find here in terms of the original transport coefficients, contrasting with their representation in terms of frame-invariant quantities in Eqs. (120) and (133). The quantities $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, and $\mathscr{E}$ are explicitly defined in the following equations:

$$\mathscr{A} = \theta_s^{(2)} + \varepsilon_\epsilon^{(2)} + \nu_n^{(2)} - \frac{1}{h}\left(\theta_\epsilon^{(1)}\varepsilon_\Theta^{(1)} + \theta_n^{(1)}\nu_\Theta^{(1)}\right), \tag{C.37}$$

$$\mathscr{B} = \mathscr{D}\left[\Gamma - v_s^2\left(\frac{\varepsilon_\Theta^{(1)}}{h}\right)\right] - h\left[\gamma_\epsilon^{(2)} + \vartheta_\epsilon^{(2)} + \frac{n}{h}\left(\gamma_n^{(2)} + \vartheta_n^{(2)}\right)\right]$$
$$+ \left(\theta_\epsilon^{(1)} + \frac{n}{h}\theta_n^{(1)}\right)\left(\lambda_s^{(1)} + \vartheta_\Theta^{(1)}\right) - \sigma\left(\alpha_s^{(2)}\beta_n + \alpha_n^{(1)}\lambda_s^{(1)}\right), \tag{C.38}$$

$$\mathscr{C} = \sigma\left(\beta_n\alpha_\epsilon^{(3)} - \beta_\epsilon\alpha_n^{(3)}\right) + \sigma h\left(\alpha_n^{(1)}\gamma_\epsilon^{(2)} - \alpha_\epsilon^{(1)}\gamma_n^{(2)}\right)$$
$$+ \sigma\alpha_s^{(2)}\left(\beta_\epsilon\theta_n^{(1)} - \beta_n\theta_\epsilon^{(1)}\right) + \sigma\lambda_s^{(1)}\left(\alpha_\epsilon^{(1)}\theta_n^{(1)} - \alpha_n^{(1)}\theta_\epsilon^{(1)}\right), \tag{C.39}$$

$$\mathscr{E} = -\frac{1}{h}\left(\beta_\epsilon\nu_n^{(2)} - \beta_n\nu_\epsilon^{(2)}\right)\varepsilon_\Theta^{(1)} + \frac{1}{h}\left(\beta_\epsilon\varepsilon_n^{(2)} - \beta_n\varepsilon_\epsilon^{(2)}\right)\nu_\Theta^{(1)}$$
$$+ \gamma_\epsilon^{(2)}\varepsilon_\Theta^{(1)} + \gamma_n^{(2)}\nu_\Theta^{(1)} - \left(\varepsilon_\epsilon^{(2)} + \nu_n^{(2)}\right)\lambda_s^{(1)} + \xi_s^{(3)} - \vartheta_\Theta^{(3)}$$
$$+ \sigma\left(\alpha_n^{(1)}\theta_\epsilon^{(1)} - \alpha_\epsilon^{(1)}\theta_n^{(1)}\right)\left(\frac{\lambda_s^{(1)}}{v_s^2} + \frac{\varepsilon_\Theta^{(1)}}{h}\right) + \frac{\sigma\beta_n}{v_s^2}\Big[\alpha_s^{(2)}\theta_\epsilon^{(1)}$$
$$- \alpha_\epsilon^{(1)}\theta_s^{(2)} - \alpha_\epsilon^{(3)} + \frac{n}{h}\left(\alpha_s^{(2)}\theta_n^{(1)} - \alpha_n^{(1)}\theta_s^{(2)} - \alpha_n^{(3)}\right)\Big]$$
$$- \frac{\sigma h}{v_s^2}\left(\alpha_n^{(1)}\gamma_\epsilon^{(2)} - \alpha_\epsilon^{(1)}\gamma_n^{(2)}\right) - \sigma\left(\alpha_n^{(1)}\varepsilon_\epsilon^{(2)} - \alpha_\epsilon^{(1)}\varepsilon_n^{(2)}\right), \tag{C.40}$$

with the various α and ϑ defined in terms of the conductivity σ and other transport coefficients as

$$\sigma\alpha_\#^{(i)} = -\left(\kappa_\#^{(i)} - \frac{n}{h}\theta_\#^{(i)}\right), \qquad \vartheta_\#^{(i)} = \frac{\beta_\epsilon\varepsilon_\#^{(i)} + \beta_n\nu_\#^{(i)}}{h}, \tag{C.41}$$

where the symbol $\#$ in the subscripts represents s, ϵ, n or Θ.

The foregoing dispersion relations involve the transport coefficients associated with the densities ϵ and n, in addition to those corresponding to σ and Θ. To express these relations in terms of the coefficients associated with the thermodynamic variables T and μ, which are frequently utilized in gradient expansions and feature in the list of structures in Section 2, it is essential to consider the subsequent relationships:

$$\delta T = \left(\frac{\partial T}{\partial\epsilon}\right)_n\delta\epsilon + \left(\frac{\partial T}{\partial n}\right)_\epsilon\delta n, \tag{C.42}$$

$$\delta\mu = \left(\frac{\partial\mu}{\partial\epsilon}\right)_n\delta\epsilon + \left(\frac{\partial\mu}{\partial n}\right)_\epsilon\delta n. \tag{C.43}$$

Consequently, transport coefficients such as $\varphi_\epsilon^{(i)}$ and $\varphi_n^{(i)}$ can be expressed in terms of $\varphi_T^{(i)}$ and $\varphi_\mu^{(i)}$ as follows:

$$\varphi_\epsilon^{(i)} = \left(\frac{\partial T}{\partial\epsilon}\right)_n\varphi_T^{(i)} + \left(\frac{\partial\mu}{\partial\epsilon}\right)_n\varphi_\mu^{(i)}, \tag{C.44}$$

$$\varphi_n^{(i)} = \left(\frac{\partial T}{\partial n}\right)_\epsilon\varphi_T^{(i)} + \left(\frac{\partial\mu}{\partial n}\right)_\epsilon\varphi_\mu^{(i)}, \tag{C.45}$$

where $i = 1, 2$ or 3, contingent upon the order of the expansion.

After extensive and intricate computations involving numerous combinations and simplifications, and taking into account the definitions provided in Eqs. (33), (36), (54), and (72), together with relations (C.44) and (C.45), we arrive at the following results:

$$\mathscr{D}\left[v_s^2(\mathscr{A} - \theta_s^{(2)}) + \mathscr{B}\right] + \mathscr{C} = -\mathscr{Y} + v_s^2\phi_n^{(3)},$$
$$\mathscr{B} = -\mathscr{X}, \quad v_s^2\left(\Gamma\mathscr{A} + \mathscr{E}\right) - \mathscr{D}\mathscr{B} = \mathscr{Y} + v_s^2\chi_s^{(3)}. \tag{C.46}$$

These equations imply that the dispersion relations (C.34) and (C.35) are equal to those presented in Section 4, specifically Eqs. (131) and (132).