

Constraining $f(\mathcal{R})$ gravity by Pulsar *SAX J1748.9-2021* observations: Supplementary material

Gamal G. L. Nashed*

*Centre for Theoretical Physics, The British University,
P.O. Box 43, El Sherouk City, Cairo 11837, Egypt.*

Salvatore Capozziello†

*Dipartimento di Fisica SE. Pancini, Università di Napoli Federico II
Complesso Universitario di Monte Sant'Angelo, Edificio G, Via Cinthia, I-80126, Napoli, Italy,
Scuola Superiore Meridionale, Largo S. Marcellino 10, I-80138, Napoli, Italy, and
Istituto Nazionale di Fisica Nucleare (INFN), Sez. di Napoli
Complesso Universitario di Monte Sant'Angelo, Edificio G, Via Cinthia, I-80126, Napoli, Italy.
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We report here, for the interested reader, the various components of the energy-momentum tensor and the form of them after substituting the KB ansatz. Moreover, we report the gradient components of the energy-momentum tensor.

* nashed@bue.edu.eg

† capozziello@na.infn.it

I. THE COMPONENTS OF THE ENERGY MOMENTUM TENSOR

The components of the energy-momentum tensor are given as:

$$\begin{aligned}
 \rho \equiv \frac{1}{r^8 \zeta^2 \kappa} e^{\frac{1}{2r^2}} & \left\{ -\zeta(2\mu''r^2 + \mu'^2r^2 + (-\nu'r^2 + 4r)\mu' - 4\nu'r + 4)e^{-\nu} + 4\zeta - 8\nu r^2 \right\} \left\{ -\zeta \left[\zeta r^2 e^\nu \mu'' - 2(\zeta + r^2)e^{2\nu} + \frac{\zeta}{2}(\mu'^2r^2 + (4r - \nu'r^2)\mu' \right. \right. \\
 & + 4 - 4\nu'r) e^\nu \left. \right] r^6 \mu'''' + \left(\zeta r^2 \mu'' + \frac{1}{2}\zeta \mu'^2 r^2 - \frac{1}{2}r\zeta(\nu'r - 4)\mu' - 2\zeta\nu'r - (2\zeta + 3r^2)e^\nu + 2\zeta \right) \zeta^2 r^6 \mu''''^2 + 2 \left[\left(2 - \frac{3}{2}\nu'r \right. \right. \\
 & + \mu'r) \zeta^2 r^4 \mu''^2 - \frac{\zeta r^3 \mu''}{2} \left[r\zeta(\mu'r + 4)\nu'' - \zeta r^2 \mu'^3 + \left(\frac{7}{2}\zeta r^2 \nu' - 6\zeta r \right) \mu'^2 + \left(-\frac{5}{2}\zeta r^2 \nu'^2 + 16\zeta \nu'r + (4\zeta + 7r^2)e^\nu - 8\zeta \right) \mu' \right. \\
 & - 10\zeta r \nu'^2 + (14\zeta - 6(\zeta + 2r^2)e^\nu)\nu' + 16re^\nu \left. \right] - \frac{1}{4}(\mu'r + 4)\zeta(\zeta \mu'^2 r^2 - r\zeta(\nu'r - 4)\mu' - 4\zeta \nu'r - (4\zeta + 6r^2)e^\nu + 4\zeta)r^2 \nu'' \\
 & + (r^3(r^2 + \zeta)\mu' + (-3r^5 - 3\zeta r^3)\nu' + 4r^4 - 8\zeta r^2 - 8\zeta^2)e^{2\nu} - \frac{1}{4} \left[\zeta \mu'^4 r^5 \nu' + (8\zeta r^4 \nu' - 2\zeta r^5 \nu'^2 + r^3(e^\nu r^2 + 4\zeta))(\mu')^3 \right. \\
 & + \left(\zeta \nu'^3 r^3 - 16\zeta r^2 \nu'^2 - 10r \left(\left(r^2 + \frac{2}{5}\zeta \right) e^\nu - \frac{8}{5}\zeta \right) \nu' + 8(r^2 - \zeta)e^\nu + 24\zeta \right) r^2 \mu'^2 + 9 \left[\frac{8}{9}\zeta \nu'^3 r^3 + \left(\left(r^2 + \frac{4}{9}\zeta \right) e^\nu - 4\zeta \right) r^2 \right. \\
 & (\nu')^2 - \frac{44}{9}((2/11\zeta + r^2)e^\nu + 2/11\zeta)r\nu' - \left(\frac{16}{3}\zeta + \frac{4}{9}r^2 \right) e^\nu + \frac{16}{3}\zeta \left. \right] r\mu' + 16\zeta(\nu')^3 r^3 + 36 \left(\left(r^2 + \frac{4}{9}\zeta \right) e^\nu - \frac{4}{9}\zeta \right) r^2 \nu'^2 \\
 & - 28r \left(\left(-\frac{8}{7}\zeta + r^2 \right) e^\nu + \frac{8}{7}\zeta \right) \nu' + (-32r^2 - 64\zeta)e^\nu + 32\zeta \left. \right] \zeta r^3 \mu'''' + \frac{1}{2}(\mu'r + 4)\zeta r^5 \mu'''' \left[\zeta r^2 e^\nu \mu'' - 2(\zeta + r^2)e^{2\nu} \right. \\
 & + \frac{1}{2}\zeta(\mu'^2 r^2 + (-\nu')r^2 + 4r)\mu' + 4 - 4(\nu')r) e^\nu \left. \right] + \left(\zeta \mu'^2 r^2 + (4\zeta r - 3\zeta r^2 \nu')\mu' + \frac{9}{4}\zeta r^2 \nu'^2 + 4\zeta - e^\nu r^2 - 6\zeta \nu'r \right) \zeta^2 r^6 \mu''^3 \\
 & - \left[\left(\zeta \mu'^2 r^2 - \frac{3}{2}r\zeta(\nu'r - 4)\mu' - 6\zeta(\nu')r - 2e^\nu r^2 + 8\zeta \right) \zeta r^2 \nu'' - \left(2\zeta r^2 - \frac{3}{2}r^4 \right) e^{2\nu} - \frac{1}{2}\zeta \left[\zeta \mu'^4 r^4 + \left(8\zeta r^3 - 6\zeta r^4 \nu' \right) \mu'^3 \right. \right. \\
 & + \left(\frac{41}{4}\zeta r^4 \nu'^2 - 38\zeta \nu'r^3 - 7 \left(\left(r^2 + \frac{4}{7}\zeta \right) e^\nu - \frac{16}{7}\zeta \right) r^2 \right) \mu'^2 - \frac{21}{4} \left[\zeta \nu'^3 r^2 - \frac{68}{7}\zeta r \nu'^2 + \left(\left(-\frac{16}{7}\zeta - \frac{32}{7}r^2 \right) e^\nu + \frac{80}{7}\zeta \right) \nu' \right. \\
 & + \frac{128}{21}re^\nu \left. \right] r^2 \mu' - 21\zeta \nu'^3 r^3 + ((-9\zeta r^2 - 19r^4)e^\nu + 49\zeta r^2)\nu'^2 + (-16\zeta r + 56r^3 e^\nu)\nu' + (16\zeta - 28r^2)e^\nu - 16\zeta \left. \right] \zeta r^4 \mu''^2 \\
 & + \frac{1}{4} \left[r^4 \zeta^2 (\mu'r + 4)^2 \nu''^2 - 2 \left[8(\zeta r + r^3)e^{2\nu} + \left(\zeta r^3 \mu'^4 + \left(10\zeta r^2 - \frac{7}{2}\zeta \nu'r^3 \right) \mu'^3 + \frac{5\mu'^2 r}{2} \left[\zeta r^2 \nu'^2 - 12\zeta \nu'r + \frac{64}{5}\zeta - e^\nu \left(\frac{18}{5}r^2 \right. \right. \right. \right. \right. \\
 & + \frac{8}{5}\zeta \left. \right) \right] + \left(20\zeta r^2 \nu'^2 + \left(\left(6\zeta r + \frac{29}{2}r^3 \right) e^\nu - 78\zeta r \right) \nu' - (16\zeta + 50r^2)e^\nu + 32\zeta \right) \mu' + 40\zeta r \nu'^2 + \left[(58r^2 + 24\zeta)e^\nu - 56\zeta \right] \nu' \\
 & - 60re^\nu \left. \right] \zeta \left. \right] r^3 \nu'' + \left[-2\mu'^2 r^6 + ((-18r^6 - 20\zeta r^4)\nu' - 64r\zeta^2 + 8r^5 - 80\zeta r^3)\mu' + (22\zeta r^4 + 22r^6)\nu'^2 + \left[96r\zeta^2 - 60r^5 \right. \right. \\
 & + 88\zeta r^3 \left. \right] \nu' - 64\zeta^2 - 176\zeta r^2 - 4r^4 \left. \right] e^{2\nu} + 4r^4 e^{3\nu} - 2\zeta \left\{ \zeta r^6 \nu' \mu'^5 - 4\zeta \left(-1 + r^2(\nu')^2 - \frac{5}{2}\nu'r \right) r^4 \mu'^4 + 5 \left[\zeta \nu'^3 r^3 - \frac{36}{5}\zeta r^2 \nu'^2 \right. \right. \\
 & - \frac{17}{10}r \left(\left(\frac{8}{17}\zeta + r^2 \right) e^\nu - \frac{44}{17}\zeta \right) \nu' + \left(\frac{2}{5}r^2 - \frac{8}{5}\zeta \right) e^\nu + \frac{32}{5}\zeta \left. \right] r^3 \mu'^3 - 2 \left[\zeta r^4 \nu'^4 - 21\zeta \nu'^3 r^3 + \left[48\zeta r^2 - \left(5\zeta r^2 + \frac{87}{8}r^4 \right) e^\nu \right] \nu'^2 \right. \\
 & + ((29r^3 - 2\zeta r)e^\nu + 18\zeta r)\nu' + (32\zeta + 8r^2)e^\nu - 44\zeta \left. \right] r^2 (\mu')^2 - \frac{53}{4}r \left[\frac{64}{53}\zeta r^4 \nu'^4 + \left(\left(\frac{24}{53}\zeta + r^2 \right) e^\nu - \frac{408}{53}\zeta \right) r^3 \nu'^3 \right. \\
 & - \frac{436}{53} \left(\left(\frac{28}{109}\zeta + r^2 \right) e^\nu - \frac{52}{109}\zeta \right) r^2 \nu'^2 + \frac{240}{53} \left((-2\zeta + r^2)e^\nu + \frac{38}{15}\zeta \right) r\nu' + \left(\frac{352}{53}r^2 + \frac{512}{53}\zeta \right) e^\nu - \frac{384}{53}\zeta \left. \right] \mu' - 32\zeta r^4 \nu'^4 \\
 & + (56\zeta r^3 - (24\zeta r^3 + 53r^5)e^\nu)\nu'^3 + ((95r^4 - 48\zeta r^2)e^\nu + 48\zeta r^2)\nu'^2 + 52 \left(\left(\frac{40}{13}\zeta + r^2 \right) e^\nu - \frac{28}{13}\zeta \right) r\nu' - (88r^2 + 64\zeta)e^\nu \\
 & + 32\zeta \left. \right\} \left. \right] \zeta r^2 \mu'' + \frac{1}{8}(\mu'r + 4)^2 \zeta^2 (\zeta \mu'^2 r^2 - r\zeta(\nu'r - 4)\mu' - 4\zeta \nu'r - (4\zeta + 6r^2)e^\nu + 4\zeta)r^4 \nu'^2 + \frac{1}{4} \left[\left((-4\zeta r^4 - 4r^6)\mu'^2 \right. \right. \\
 & + ((14\zeta r^4 + 14r^6)\nu' + 32r\zeta^2 - 24r^5 + 24\zeta r^3)\mu' + (56r^5 + 56\zeta r^3)\nu' + 176\zeta r^2 - 16r^4 + 128\zeta^2 \left. \right] e^{2\nu} + \zeta \left(\zeta r^6 \nu' \mu'^5 \right. \\
 & - 2 \left(-6\zeta \nu'r - \frac{1}{2}e^\nu r^2 - 2\zeta + \zeta r^2 \nu'^2 \right) r^4 \mu'^4 + \left[\zeta \nu'^3 r^3 - 24\zeta r^2 \nu'^2 - 2\frac{1}{2}r \left(\left(\frac{8}{21}\zeta + r^2 \right) e^\nu - \frac{32}{7}\zeta \right) \nu' + (-8\zeta + 10r^2)e^\nu \right.
 \end{aligned}$$

$$\begin{aligned}
& + 40\zeta \left[r^3 \mu'^3 + 19/2 \left(\frac{24}{19} \zeta \nu'^3 r^3 + r^2 \left(\left(r^2 + \frac{8}{19} \zeta \right) e^\nu - \frac{200}{19} \zeta \right) \nu'^2 - \frac{172}{19} \left(\left(r^2 + \frac{12}{43} \zeta \right) e^\nu - \frac{28}{43} \zeta \right) r \nu' + \left(\frac{16}{19} r^2 \right. \right. \right. \\
& - \left. \frac{160}{19} \zeta \right) e^\nu + \frac{288}{19} \zeta \left. \right) r^2 \mu'^2 + 76 \left[\frac{12}{19} \zeta \nu'^3 r^3 + \left(\left(r^2 + \frac{8}{19} \zeta \right) e^\nu - \frac{40}{19} \zeta \right) r^2 \nu'^2 - \frac{97}{38} \left(e^\nu r^2 + \frac{32}{97} \zeta \right) r \nu' - \left[\frac{26}{19} r^2 + \frac{64}{19} \zeta \right] \right. \\
& e^\nu + \frac{56}{19} \zeta \left. \right] r \mu' + 64 \zeta (\nu')^3 r^3 + 152 \left(\left(r^2 + \frac{8}{19} \zeta \right) e^\nu - \frac{8}{19} \zeta \right) r^2 (\nu')^2 - 72 \left[\left(-\frac{16}{9} \zeta + r^2 \right) e^\nu + \frac{16}{9} \zeta \right] r \nu' \\
& + \left(-176 r^2 - 256 \zeta \right) e^\nu + 128 \zeta \left. \right] \zeta r^2 \nu'' - \frac{1}{8} \left[\mu'^4 r^8 + (8 r^7 - 2 r^8 \nu') \mu'^3 + \left[(-12 r^6 \zeta - 11 r^8) \nu'^2 + (4 r^7 - 64 \zeta^2 r^3 - 88 r^5 \zeta) \nu' \right. \right. \\
& - 16 \zeta r^4 - 64 \zeta^2 r^2 + 20 r^6 \left. \right] \mu'^2 + 12 \left[r^5 (r^2 + \zeta) \nu'^3 + (10/3 \zeta r^4 + 16/3 \zeta^2 r^2 - 5 r^6) \nu'^2 + \left(-3 r^5 - \frac{104}{3} \zeta r^3 - 16 r \zeta^2 \right) \nu' \right. \\
& - \frac{128}{3} \zeta^2 + 4/3 r^4 - \frac{112}{3} \zeta r^2 \left. \right] r \mu' + (48 r^7 + 48 r^5 \zeta) \nu'^3 + (-32 r^6 + 256 \zeta^2 r^2 + 384 \zeta r^4) \nu'^2 + (-96 r^5 - 64 \zeta r^3 + 256 r \zeta^2) \nu' \\
& - 64 r^4 - 896 \zeta r^2 - 768 \zeta^2 \left. \right] \zeta e^{2\nu} + \left(\frac{1}{2} \zeta \mu'^2 r^6 - \frac{1}{2} r^5 \zeta (\nu' r - 4) \mu' + r^3 (r^4 - 4 \zeta r^2 - 4 \zeta^2) \nu' - 56 \zeta^2 r^2 - 8 \zeta r^4 - r^6 - 32 \zeta^3 \right) e^{3\nu} \\
& + e^{4\nu} r^6 + 1/8 \zeta^2 \left[\zeta \nu'^2 \mu'^6 r^8 - 3 \zeta \nu' r^6 \left(-\frac{8}{3} - 4 \nu' r + r^2 \nu'^2 \right) (\mu')^5 + 3 \left[\zeta r^4 \nu'^4 - 12 \zeta \nu'^3 r^3 - 3 \left[\left(r^2 + \frac{4}{9} \zeta \right) e^\nu - 4 \zeta \right] r^2 \nu'^2 \right. \right. \\
& + 4/3 \left((r^2 - 4 \zeta) e^\nu + 20 \zeta \right) r \nu' + \frac{16}{3} \zeta \left. \right] r^4 \mu'^4 - r^3 \left[\zeta \nu'^5 r^5 - 36 \zeta r^4 \nu'^4 - 18 \left(\left(r^2 + \frac{4}{9} \zeta \right) e^\nu - 8 \zeta \right) r^3 \nu'^3 + (80 r^4 e^\nu + 64 \zeta r^2) \nu'^2 \right. \\
& + 36 r \left(\left(\frac{40}{9} \zeta + r^2 \right) e^\nu - \frac{68}{9} \zeta \right) \nu' + 64 \zeta (e^\nu - 2) \left. \right] \mu'^3 - 9 \left[\frac{4}{3} \zeta \nu'^5 r^5 + \left(\left(r^2 + \frac{4}{9} \zeta \right) e^\nu - \frac{148}{9} \zeta \right) r^4 \nu'^4 - \frac{148}{9} \left[\left(\frac{12}{37} \zeta + r^2 \right) e^\nu \right. \right. \\
& - \frac{44}{37} \zeta \left. \right] r^3 \nu'^3 + 16 \left(\left(r^2 - \frac{14}{9} \zeta \right) e^\nu + \frac{10}{3} \zeta \right) r^2 \nu'^2 + \frac{104}{3} \left(\left(r^2 + \frac{56}{39} \zeta \right) e^\nu - \frac{40}{39} \zeta \right) r \nu' + \left(\frac{448}{9} \zeta + \frac{112}{9} r^2 \right) e^\nu - \frac{128}{3} \zeta \left. \right] r^2 \mu'^2 \\
& - 72 \left[\frac{2}{3} \zeta \nu'^5 r^5 + \left((r^2 + 4/9 \zeta) e^\nu - \frac{28}{9} \zeta \right) r^4 \nu'^4 + \left(-9/2 r^5 e^\nu - \frac{16}{9} \zeta r^3 \right) \nu'^3 - \frac{29}{9} \left(\left(\frac{96}{29} \zeta + r^2 \right) e^\nu - \frac{88}{29} \zeta \right) r^2 \nu'^2 \right. \\
& + \frac{20}{3} \left(\left(\frac{4}{15} \zeta + r^2 \right) e^\nu + 2/15 \zeta \right) r \nu' + \left(\frac{56}{9} r^2 + \frac{128}{9} \zeta \right) e^\nu - \frac{64}{9} \zeta \left. \right] r \mu' - 64 \zeta \nu'^5 r^5 - 144 \left((r^2 + 4/9 \zeta) e^\nu - 4/9 \zeta \right) r^4 \nu'^4 \\
& + 80 \left(\left(r^2 - \frac{16}{5} \zeta \right) e^\nu + \frac{16}{5} \zeta \right) r^3 (\nu')^3 + 448 \left(\left(r^2 + \frac{8}{7} \zeta \right) e^\nu - 4/7 \zeta \right) r^2 \nu'^2 - 32 r \left((-16 \zeta + r^2) e^\nu + 8 \zeta \right) \nu' \\
& + \left. \left(-768 \zeta - 448 r^2 \right) e^\nu + 256 \zeta \right] \Big\}, \tag{1}
\end{aligned}$$

$$\begin{aligned}
p_r \equiv & \frac{-1}{8r^6\kappa} e^{\frac{-(2r^2\mu''+\mu'^2r^2+(4r-r^2\nu')\mu'+4-4\nu'r)\zeta e^{-\nu}+4\zeta-6\nu r^2}{2r^2}} \left\{ -2r^3(\mu'r+4)\zeta \left[2\zeta r^2\mu'' + \zeta\mu'^2r^2 - r\zeta(-4+(\nu')r)\mu' - 4\zeta\nu'r + 4\zeta \right. \right. \\
& - 4(r^2+\zeta)e^\nu \Big] \mu''' - 4r^4\zeta(\zeta\mu'^2r^2 - 3/2r\zeta(\nu'r-4)\mu' + e^\nu r^2 - 6\zeta\nu'r + 8\zeta)\mu''^2 - 2r^3\zeta \left[-\zeta r(\mu'r+4)^2\nu'' - 4re^{2\nu} + \zeta r^3\mu'^4 \right. \\
& + (10\zeta r^2 - 7/2\nu'r^3\zeta)\mu'^3 + 5/2r \left(\zeta r^2(\nu')^2 - 12\zeta(\nu')r + (-4/5r^2 - 8/5\zeta)e^\nu + \frac{64}{5}\zeta \right) \mu'^2 + \left\{ 20\zeta r^2\nu'^2 + 4r \left[(r^2 + 3/2\zeta)e^\nu \right. \right. \\
& - \frac{39}{2}\zeta \Big] \nu' + (-16\zeta - 20r^2)e^\nu + 32\zeta \Big\} \mu' + 40\zeta r(\nu')^2 + ((16r^2 + 24\zeta)e^\nu - 56\zeta)\nu' - 28e^\nu r \Big] \mu'' + r^2(\mu'r+4)^2\zeta \\
& (\zeta\mu'^2r^2 - r\zeta(-4+\nu'r)\mu' - 4\zeta\nu'r + (-4r^2 - 4\zeta)e^\nu + 4\zeta)\nu'' + \left[4\zeta\mu'^2r^4 + (32r^3\zeta + 32r\zeta^2 - 4r^4\zeta\nu' - 8r^5)\mu' - 8r^4 + 128\zeta^2 \right. \\
& - 16\nu'r^3\zeta + 128\zeta r^2 \Big] e^{2\nu} + 8e^{3\nu}r^4 + \left[\zeta\mu'^5\nu'r^6 - 2r^4(-6\zeta(\nu')r - 2\zeta + \zeta r^2\nu'^2 + 1/2e^\nu r^2)\mu'^4 + r^3 \left(\zeta\nu'^3r^3 - 24\zeta r^2\nu'^2 \right. \right. \\
& - 2r((r^2+2\zeta)e^\nu - 24\zeta)\nu' - 4(2\zeta+r^2)e^\nu + 40\zeta \Big) \mu'^3 + 3r^2 \left(4\zeta\nu'^3r^3 + r^2 \left((4/3\zeta+r^2)e^\nu - \frac{100}{3}\zeta \right) \nu'^2 \right. \\
& - \frac{20}{3}r \left(\left(\frac{6}{5}\zeta + r^2 \right) e^\nu - \frac{14}{5}\zeta \right) \nu' + \left(-\frac{20}{3}r^2 - \frac{80}{3}\zeta \right) e^\nu + 48\zeta \Big) \mu'^2 + 24r \left[2\zeta\nu'^3r^3 + r^2 \left((4/3\zeta+r^2)e^\nu - \frac{20}{3}\zeta \right) \nu'^2 - \left(\frac{8}{3}r\zeta \right. \right. \\
& + \frac{11}{6}r^3e^\nu \Big) \nu' - \left(\frac{32}{3}\zeta + 4r^2 \right) e^\nu + \frac{28}{3}\zeta \Big] \mu' + 64\zeta\nu'^3r^3 + 48r^2 \left(\left(\frac{4}{3}\zeta + r^2 \right) e^\nu - \frac{4}{3}\zeta \right) \nu'^2 + 16r((8\zeta+r^2)e^\nu - 8\zeta)\nu' \\
& \left. \left. - 128(2\zeta+r^2)e^\nu + 128\zeta \right] \zeta \right\}, \tag{2}
\end{aligned}$$

$$\begin{aligned}
p_t \equiv & -\frac{1}{8r^8\kappa} e^{-\frac{(2r^2\mu''+\mu'^2r^2+(-r^2\nu'+4r)\mu'+4-4\nu'r)\zeta e^{-\nu}+4\zeta-8\nu r^2}{2r^2}} \left\{ -4r^6 \left[2\zeta r^2 e^\nu \mu'' - 4(r^2 + \zeta) e^{2\nu} + e^\nu \zeta \left(\mu'^2 r^2 + (4r - r^2 \nu') \mu' + 4 \right. \right. \right. \\
& \left. \left. \left. - 4\nu' r \right) \right] \zeta \mu'''' + 4r^6 \zeta^2 (2\zeta r^2 \mu'' + \zeta \mu'^2 r^2 - r\zeta (\nu' r - 4) \mu' - 4\zeta \nu' r - (6r^2 + 4\zeta) e^\nu + 4\zeta) \mu'''' - 4r^3 \left[-4r^4 \zeta^2 \left(2 - \frac{3}{2} \nu' r \right. \right. \right. \\
& \left. \left. \left. + \mu' r \right) \mu''^2 - 2r^3 \left[-r\zeta ((\mu') r + 4) \nu'' + \zeta r^2 \mu'^3 + \left(6r\zeta - \frac{7}{2} \zeta \nu' r^2 \right) \mu'^2 + \left(\frac{5}{2} \zeta r^2 \nu'^2 - 16\zeta \nu' r - \left(4\zeta + \frac{15}{2} r^2 \right) e^\nu + 8\zeta \right) \mu' \right. \right. \right. \\
& \left. \left. \left. + 10\zeta r \nu'^2 + ((12r^2 + 6\zeta) e^\nu - 14\zeta) \nu' - 15e^\nu r \right] \zeta \mu' + r^2 (\mu' r + 4) \zeta (\zeta \mu'^2 r^2 - r\zeta (\nu' r - 4) \mu' - 4\zeta \nu' r - (6r^2 + 4\zeta) e^\nu + 4\zeta) \nu'' \right. \right. \\
& \left. \left. + ((-6r^5 - 6r^3 \zeta) \mu' + (12r^3 \zeta + 12r^5) \nu' - 12r^4 + 32\zeta^2 + 36\zeta r^2) e^{2\nu} + \left[\zeta \mu'^4 r^5 \nu' - 2r^3 (\zeta r^2 \nu'^2 - 2\zeta - 3/4 e^\nu r^2 - 4\zeta \nu' r) \mu'^3 \right. \right. \right. \\
& \left. \left. \left. + r^2 \left(\zeta \nu'^3 r^3 - 16\zeta r^2 \nu'^2 - \frac{21}{2} r \left(\left(\frac{8}{21} \zeta + r^2 \right) e^\nu - \frac{32}{21} \zeta \right) \nu' + (9r^2 - 8\zeta) e^\nu + 24\zeta \right) \mu'^2 + 9r \left(\frac{8}{9} \zeta \nu'^3 r^3 + r^2 \left[\left(r^2 + \frac{4}{9} \zeta \right) e^\nu \right. \right. \right. \right. \\
& \left. \left. \left. - 4\zeta \right] \nu'^2 - 5r \left(\left(\frac{8}{45} \zeta + r^2 \right) e^\nu + \frac{8}{45} \zeta \right) \nu' + (-16/3\zeta - 2/3r^2) e^\nu + 16/3\zeta \right) \mu' + 16\zeta \nu'^3 r^3 + 36r^2 \left(\left(r^2 + \frac{4}{9} \zeta \right) e^\nu - \frac{4}{9} \zeta \right) \nu'^2 \right. \right. \\
& \left. \left. - 24r \left(\left(r^2 - \frac{4}{3} \zeta \right) e^\nu + \frac{4}{3} \zeta \right) \nu' - (64\zeta + 36r^2) e^\nu + 32\zeta \right] \zeta \mu'''' + 2r^5 \left[2\zeta r^2 e^\nu \mu'' - 4(r^2 + \zeta) e^{2\nu} + e^\nu \zeta \left\{ \mu'^2 r^2 + 4 - 4\nu' r \right. \right. \right. \\
& \left. \left. \left. + (4r - r^2 \nu') \mu' \right\} \right] (\mu' r + 4) \zeta \nu'''' + 8r^6 \zeta^2 \left(\zeta (\mu')^2 r^2 + (4r\zeta - 3\zeta \nu' r^2) \mu' + \frac{9}{4} \zeta r^2 (\nu')^2 + 4\zeta - e^\nu r^2 - 6\zeta \nu' r \right) \mu'^3 \right. \\
& \left. + 4r^4 \left[-2r^2 \left(\zeta (\mu')^2 r^2 - 3/2r\zeta (-4 + \nu' r) \mu' - 2e^\nu r^2 + 8\zeta - 6\zeta \nu' r \right) \zeta \nu'' + (4\zeta r^2 + 4r^4) e^{2\nu} + \left[\zeta \mu'^4 r^4 + (8r^3 \zeta - 6r^4 \zeta \nu') \mu'^3 \right. \right. \right. \\
& \left. \left. \left. + \left(\frac{41}{4} \zeta \nu'^2 r^4 - 38\nu' r^3 \zeta - 8r^2 ((r^2 + 1/2\zeta) e^\nu - 2\zeta) \right) \mu'^2 - \frac{21}{4} r^2 \left(\zeta \nu'^3 r^2 - \frac{68}{7} \zeta r \nu'^2 + \left(\left(-\frac{16}{7} \zeta - \frac{34}{7} r^2 \right) e^\nu + \frac{80}{7} \zeta \right) \nu' \right. \right. \right. \\
& \left. \left. \left. + \frac{128}{21} e^\nu r \right) \mu' - 21\zeta (\nu')^3 r^3 + ((-19r^4 - 9\zeta r^2) e^\nu + 49\zeta r^2) \nu'^2 + (-16r\zeta + 53r^3 e^\nu) \nu' + (16\zeta - 24r^2) e^\nu - 16\zeta \right] \zeta \mu''^2 \right. \\
& \left. - 4r^2 \left\{ -\frac{1}{2} \zeta^3 r^4 (\mu' r + 4)^2 \nu'^2 + r^3 \left[(8r\zeta + 8r^3) e^{2\nu} + \left(\zeta r^3 (\mu')^4 + (10\zeta r^2 - 7/2\nu' r^3 \zeta) \mu'^3 + 5/2r \left(\zeta r^2 \nu'^2 - 12\zeta \nu' r \right. \right. \right. \right. \right. \\
& \left. \left. \left. + \left(-8/5\zeta - \frac{19}{5} r^2 \right) e^\nu + \frac{64}{5} \zeta \right) \mu'^2 + \left(20\zeta r^2 \nu'^2 + \left(\left(\frac{29}{2} r^3 + 6r\zeta \right) e^\nu - 78r\zeta \right) \nu' + (-51r^2 - 16\zeta) e^\nu + 32\zeta \right) \mu' + 40\zeta r \nu'^2 \right. \right. \right. \\
& \left. \left. \left. + ((24\zeta + 58r^2) e^\nu - 56\zeta) \nu' - 56e^\nu r \right) \zeta \right] \zeta \nu'' - 2 \left[r^4 (r^2 + \zeta) \mu'^2 + (-13/2r^4 (r^2 + \zeta) \nu' + 7/2r^5 - 16r\zeta^2 - 22r^3 \zeta) \mu' \right. \right. \\
& \left. \left. + 11/2r^4 (r^2 + \zeta) \nu'^2 + \left(25r^3 \zeta - \frac{25}{2} r^5 + 24r\zeta^2 \right) \nu' - 16\zeta^2 - 44\zeta r^2 - 5r^4 \right] \zeta e^{2\nu} + r^4 (r^2 - 2\zeta) e^{3\nu} + \left[\zeta (\mu')^5 \nu' r^6 \right. \right. \\
& \left. \left. - 4r^4 \left(\zeta r^2 \nu'^2 - \frac{5}{2} \zeta \nu' r - \zeta - \frac{1}{8} e^\nu r^2 \right) \mu'^4 + 5r^3 \mu'^3 \left(\zeta \nu'^3 r^3 - \frac{36}{5} \zeta r^2 \nu'^2 - \frac{41}{20} r \left(\left(r^2 + \frac{16}{41} \zeta \right) e^\nu - \frac{88}{41} \zeta \right) \nu' + \left(\frac{4}{5} r^2 - \frac{8}{5} \zeta \right) e^\nu \right. \right. \right. \\
& \left. \left. \left. + \frac{32}{5} \zeta \right) - 2r^2 \left(\zeta \nu'^4 r^4 - 21\zeta \nu'^3 r^3 + \left(\left(-5\zeta r^2 - \frac{23}{2} r^4 \right) e^\nu + 48\zeta r^2 \right) \nu'^2 + \left(\left(\frac{125}{4} r^3 - 2r\zeta \right) e^\nu + 18r\zeta \right) \nu' + (9r^2 + 32\zeta) e^\nu \right. \right. \right. \\
& \left. \left. \left. - 44\zeta \right) (\mu')^2 - \frac{53}{4} r \left(\frac{64}{53} \zeta \nu'^4 r^4 + r^3 \left(\left(r^2 + \frac{24}{53} \zeta \right) e^\nu - \frac{408}{53} \zeta \right) (\nu')^3 - \frac{446}{53} r^2 \left(\left(\frac{56}{223} \zeta + r^2 \right) e^\nu - \frac{104}{223} \zeta \right) \nu'^2 \right. \right. \right. \\
& \left. \left. \left. + \frac{204}{53} r \left(\left(r^2 - \frac{40}{17} \zeta \right) e^\nu + \frac{152}{51} \zeta \right) \nu' + \left(\frac{384}{53} r^2 + \frac{512}{53} \zeta \right) e^\nu - \frac{384}{53} \zeta \right) \mu' - 32\zeta \nu'^4 r^4 + ((-53r^5 - 24r^3 \zeta) e^\nu + 56r^3 \zeta) \nu'^3 \right. \right. \\
& \left. \left. + ((85r^4 - 48\zeta r^2) e^\nu + 48\zeta r^2) \nu'^2 + 66r \left(\left(\frac{80}{33} \zeta + r^2 \right) e^\nu - \frac{56}{33} \zeta \right) \nu' + (-64\zeta - 88r^2) e^\nu + 32\zeta \right] \zeta^2 \right\} \mu'' + r^4 (\mu' r + 4)^2 \zeta^2 \\
& \times (\zeta \mu'^2 r^2 - r\zeta (-4 + \nu' r) \mu' - 4\zeta \nu' r + (-6r^2 - 4\zeta) e^\nu + 4\zeta) \nu'^2 + 2r^2 \left\{ [(-6r^6 - 6r^4 \zeta) \mu'^2 + \left((14r^6 + 14r^4 \zeta) \nu' - 28r^5 \right. \right. \\
& \left. \left. + 20r^3 \zeta + 32r\zeta^2 \right) \mu' + (56r^5 + 56r^3 \zeta) \nu' + 128\zeta^2 + 192\zeta r^2] e^{2\nu} + \left[\zeta \mu'^5 \nu' r^6 - 2r^4 (-6\zeta \nu' r + \zeta r^2 \nu'^2 - 2\zeta - 3/4 e^\nu r^2) \mu'^4 \right. \right. \\
& \left. \left. + r^3 (\zeta \nu'^3 r^3 - 24\zeta r^2 \nu'^2 + (48r\zeta - (11r^3 + 4r\zeta) e^\nu) \nu' + (13r^2 - 8\zeta) e^\nu + 40\zeta) \mu'^3 + \frac{19}{2} r^2 \left[\frac{24}{19} \zeta \nu'^3 r^3 + r^2 \left(\left(\frac{8}{19} \zeta + r^2 \right) e^\nu \right. \right. \right. \right. \\
& \left. \left. \left. - \frac{200}{19} \zeta \right) \nu'^2 - \frac{178}{19} r \left(\left(\frac{24}{89} \zeta + r^2 \right) e^\nu - \frac{56}{89} \zeta \right) \nu' + \left(-\frac{160}{19} \zeta + \frac{20}{19} r^2 \right) e^\nu + \frac{288}{19} \zeta \right] \mu'^2 + 76r \left[\frac{12}{19} \zeta \nu'^3 r^3 + r^2 \left(\left(\frac{8}{19} \zeta + r^2 \right) e^\nu \right. \right. \right. \\
& \left. \left. \left. - \frac{40}{19} \zeta \right) \nu'^2 - \frac{97}{38} r \left(e^\nu r^2 + \frac{32}{97} \zeta \right) \nu' + \left(-\frac{29}{19} r^2 - \frac{64}{19} \zeta \right) e^\nu + \frac{56}{19} \zeta \right] \mu' + 64\zeta \nu'^3 r^3 + 152r^2 \left(\left(\frac{8}{19} \zeta + r^2 \right) e^\nu - \frac{8}{19} \zeta \right) \nu'^2 \right. \\
& \left. \left. + \left(\left(\frac{8}{19} \zeta + r^2 \right) e^\nu - \frac{8}{19} \zeta \right) \nu'^2 \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& -56r \left(\left(r^2 - \frac{16}{7}\zeta \right) e^\nu + \frac{16}{7}\zeta \right) \nu' + (-192r^2 - 256\zeta) e^\nu + 128\zeta \Big] \zeta \Big\} \zeta \nu'' - 4\zeta \left[r^5 (r (r^2 + \zeta) \nu' + 1/2r^2 + 2\zeta) \mu'^3 - 4r^2 \right. \\
& \left(r^4 (r^2 + \zeta) \nu'^2 + \left(4r\zeta^2 - \frac{5}{4}r^5 + \frac{11}{2}r^3\zeta \right) \nu' - \frac{7}{4}r^4 - \zeta r^2 + 4\zeta^2 \right) \mu'^2 + 3r \left(r^5 (r^2 + \zeta) \nu'^3 + \left(\frac{16}{3}\zeta^2 r^2 + \frac{8}{3}r^4\zeta - \frac{35}{6}r^6 \right) \nu'^2 \right. \\
& + \left(-16r\zeta^2 - \frac{116}{3}r^3\zeta - \frac{17}{3}r^5 \right) \nu' - 40\zeta r^2 - \frac{128}{3}\zeta^2 + 2r^4 \Big) \mu' + (12r^7 + 12\zeta r^5) \nu'^3 + (104r^4\zeta + 2r^6 + 64\zeta^2 r^2) \nu'^2 \\
& + (-30r^5 + 64r\zeta^2) \nu' - 192\zeta^2 - 256\zeta r^2 - 28r^4 \Big] e^{2\nu} + \left((-2r^8 + 4r^6\zeta) \mu'^2 + 2r^3 (r^3 (r^2 - 2\zeta) \nu' + 28\zeta r^2 + 16\zeta^2 - 2r^4) \mu' \right. \\
& + (-56\zeta r^5 - 32\zeta^2 r^3 + 4r^7) \nu' - 256\zeta^3 - 512\zeta^2 r^2 - 96r^4\zeta \Big) e^{3\nu} + \left[-16r^4 e^{4\nu} + \left\{ \zeta \nu'^2 \mu'^6 r^8 - 3r^6 \nu' \left(\zeta r^2 \nu'^2 - 4\zeta \nu' r - \frac{8}{3}\zeta \right. \right. \right. \\
& \left. \left. - 1/3 e^\nu r^2 \right) \mu'^5 + 3r^4 \left(\zeta \nu'^4 r^4 - 12\zeta \nu'^3 r^3 - 11/3 r^2 \left((r^2 + 4/11\zeta) e^\nu - \frac{36}{11}\zeta \right) \nu'^2 + 10/3 r ((-8/5\zeta + r^2) e^\nu + 8\zeta) \nu' \right. \right. \\
& \left. \left. + 4/3 e^\nu r^2 + 16/3\zeta \right) \mu'^4 - r^3 \left[\zeta \nu'^5 r^5 - 36\zeta \nu'^4 r^4 + ((-19r^5 - 8r^3\zeta) e^\nu + 144r^3\zeta) \nu'^3 + (64\zeta r^2 + 92r^4 e^\nu) \nu'^2 \right. \right. \\
& \left. \left. + 36r \left(\left(\frac{40}{9}\zeta + r^2 \right) e^\nu - \frac{68}{9}\zeta \right) \nu' + (64\zeta - 16r^2) e^\nu - 128\zeta \right] \mu'^3 - 9r^2 \left[4/3 \zeta \nu'^5 r^5 + r^4 \left((r^2 + 4/9\zeta) e^\nu - \frac{148}{9}\zeta \right) \nu'^4 \right. \right. \\
& \left. \left. - \frac{154}{9} r^3 \left(\left(r^2 + \frac{24}{77}\zeta \right) e^\nu - \frac{8}{7}\zeta \right) \nu'^3 + \frac{148}{9} r^2 \left(\left(r^2 - \frac{56}{37}\zeta \right) e^\nu + \frac{120}{37}\zeta \right) \nu'^2 + \frac{352}{9} r \left(\left(r^2 + \frac{14}{11}\zeta \right) e^\nu - \frac{10}{11}\zeta \right) \nu' \right. \right. \\
& \left. \left. + \left(\frac{448}{9}\zeta + \frac{112}{9} r^2 \right) e^\nu - \frac{128}{3}\zeta \right] \mu'^2 - 72r \left[2/3 \zeta \nu'^5 r^5 + r^4 \left((r^2 + 4/9\zeta) e^\nu - \frac{28}{9}\zeta \right) \nu'^4 + \left(-9/2 r^5 e^\nu - \frac{16}{9} r^3 \zeta \right) \nu'^3 \right. \right. \\
& \left. \left. - 4r^2 \left((r^2 + 8/3\zeta) e^\nu - \frac{22}{9}\zeta \right) \nu'^2 + \frac{62}{9} r \left(\left(r^2 + \frac{8}{31}\zeta \right) e^\nu + \frac{4}{31}\zeta \right) \nu' + \left(\frac{128}{9}\zeta + \frac{64}{9} r^2 \right) e^\nu - \frac{64}{9}\zeta \right] \mu' - 64\zeta \nu'^5 r^5 \right. \\
& \left. - 144 \left((r^2 + 4/9\zeta) e^\nu - 4/9\zeta \right) r^4 \nu'^4 + 48r^3 \left((-16/3\zeta + r^2) e^\nu + 16/3\zeta \right) \nu'^3 + 480r^2 \left(\left(\frac{16}{15}\zeta + r^2 \right) e^\nu - \frac{8}{15}\zeta \right) \nu'^2 \right. \\
& \left. + 32r \left((16\zeta + r^2) e^\nu - 8\zeta \right) \nu' + (-512r^2 - 768\zeta) e^\nu + 256\zeta \right\} \zeta \Big] \zeta \Big\}, \tag{3}
\end{aligned}$$

II. THE FORM OF THE COMPONENTS OF ρ , p_r AND p_t ADOPTING THE KB ANSATZ

$$\begin{aligned}
\rho = & \frac{1}{c^2 r^8 R^{16} \kappa} \left\{ -56R^4 \left[\frac{12}{7} (R^4 - r^2 (s_0 - 2s_2) R^2 + r^4 s_0 s_2) (R^2 + r^2 (s_0 - s_2)) \left(R^6 + 2 \left(s_0 - \frac{s_2}{2} \right) r^2 R^4 + \frac{r^4}{3} (2s_0 s_2 - 2s_2^2 \right. \right. \right. \\
& + s_0^2) R^2 + \frac{r^6 s_0 s_2}{3} (s_0 - s_2) \Big) \zeta + r^2 \left(R^{12} + \frac{3}{7} r^2 (s_0 + 4s_2) R^{10} - \frac{8}{7} \left(\frac{13}{8} s_2^2 + s_0^2 - \frac{41}{8} s_0 s_2 \right) r^4 R^8 + \frac{9}{7} r^6 \left(s_0^3 - \frac{16}{3} s_0^2 s_2 \right. \right. \\
& + \frac{34}{3} s_0 s_2^2 - \frac{16}{3} s_2^3 \Big) R^6 + \frac{9}{7} r^8 \left(\frac{223}{9} s_0^2 s_2^2 + s_0^4 + 4s_2^4 - \frac{29}{3} s_0^3 s_2 - \frac{172}{9} s_2^3 s_0 \right) R^4 - \frac{20}{7} s_0 (s_0 - s_2) r^{10} s_2 \left(\frac{9}{5} s_2^2 - \frac{37}{10} s_0 s_2 \right. \\
& + s_0^2) R^2 + \frac{9r^{12}}{7} s_0^2 s_2^2 (s_0 - s_2)^2 \Big] \zeta^2 X(r) + 32 (R^4 - r^2 (s_0 - 2s_2) R^2 + r^4 s_0 s_2)^2 (R^2 + r^2 (s_0 - s_2))^2 (R^4 + 3 (s_0 r^2 R^2 \\
& - \frac{2s_2}{3}) + r^4 s_0 (s_0 - s_2)) \zeta^3 X_1(r) + (8 (12 (R^6 + r^2 s_0 R^4 - 1/3 r^4 (-7s_0 s_2 + s_0^2 + 4s_2^2) R^2 + 2/3 r^6 s_0 s_2 (s_0 - s_2)) R^2 \zeta^2 \\
& + 14r^2 \zeta (R^8 + 3/14 r^2 (s_0 + 13/3 s_2) R^6 - 4/7 (s_0 - 1/2 s_2) (s_0 - \frac{17}{4} s_2) r^4 R^4 + \frac{5}{14} r^6 s_2 (4/5 s_0 s_2 + s_0^2 - 6/5 s_2^2) R^2 \\
& + \frac{3}{14} r^8 s_0 s_2^2 (s_0 - s_2)) + (R^8 - 3/4 (s_0 - 8/3 s_2) r^2 R^6 + 1/2 (s_0^2 - \frac{41}{2} s_0 s_2 + 18s_2^2) r^4 R^4 - 3/2 r^6 (4s_0^2 s_2 - 11s_0 s_2^2 \\
& + 4s_2^3 + s_0^3) R^2 - 1/4 r^8 s_0 (s_0 - s_2) (s_0 + 3s_2) (-4s_2 + s_0)) r^4) \zeta X_2(r) + ((-32R^4 \zeta^3 + (-8s_2 r^4 R^2 - 56R^4 r^2) \zeta^2 \\
& - 8 (R^2 - r^2 (s_0 - s_2)) r^4 (1/4 r^2 s_0 + R^2) \zeta + 2R^2 s_2 r^8 - R^4 r^6) X_3(r) + R^4 X_4(r) r^6) R^4) R^8 \Big\}, \\
p_r = & \frac{1}{r^6 R^{12} \kappa} \left\{ 16R^4 \zeta \left((r^2 + 2\zeta) R^8 + 5/8 \left((6/5 s_2 + s_0) r^2 + \frac{32}{5} (s_0 - 1/4 s_2) \zeta \right) r^2 R^6 - \frac{1}{2} r^4 \left(\left(3s_2^2 + s_0^2 - \frac{27}{4} s_0 s_2 \right) r^2 \right. \right. \right. \\
& - 3 (5/3 s_0 s_2 + s_0^2 - 4/3 s_2^2) \zeta) R^4 + 2 ((-3/4 s_2 + s_0) r^2 + 5/4 (-4/5 s_2 + s_0) \zeta) s_2 r^6 s_0 R^2 + 1/8 ((s_0 + 3s_2) r^2 + 4\zeta s_2) r^8 \\
& (s_0 - s_2) s_0^2) X_2(r) - 16 (R^2 + 1/2 r^2 s_0) (R^4 - r^2 (s_0 - 2s_2) R^2 + r^4 s_0 s_2) (R^4 + 3 (-2/3 s_2 + s_0) r^2 R^2 + r^4 s_0 (s_0 - s_2)) \zeta^2 \\
& (R^2 + r^2 (s_0 - s_2)) X(r) - (((16\zeta r^2 + 16\zeta^2 - r^4) R^4 - 2r^2 (r^4 s_0 - 5 (s_0 - 2/5 s_2) \zeta r^2 - 4\zeta^2 s_0) R^2 + 2r^6 s_0 \zeta (s_0 - s_2)) X_3(r) \\
& + R^4 X_4(r) r^4) R^8 \Big\}, \\
p_t = & \frac{1}{r^8 R^{16} \kappa} \left\{ 64 \left(\left(\frac{3}{2} \zeta + r^2 \right) R^{12} + \left(\left(\frac{5}{8} s_0 + \frac{11}{8} s_2 \right) r^4 + 3\zeta r^2 s_0 \right) R^{10} - \frac{11}{8} r^4 \left(\left(s_0^2 + \frac{17}{11} s_2^2 - \frac{48}{11} s_0 s_2 \right) r^2 + \frac{8}{11} (s_0^2 \right. \right. \right. \\
& + \frac{11s_2^2}{2} - 10s_0 s_2) \zeta) R^8 + \frac{3}{4} r^6 \left(\left(15s_0 s_2^2 - \frac{19}{3} s_0^2 s_2 + s_0^3 - \frac{22}{3} s_2^3 \right) r^2 - 4 \left(s_0^3 - 16/3 s_0^2 s_2 - \frac{2}{3} s_2^3 + \frac{13}{3} s_0 s_2^2 \right) \zeta \right) R^6 \\
& + \frac{11}{8} \left(\left(-\frac{172}{11} s_2^3 s_0 + \frac{36}{11} s_2^4 + \frac{228}{11} s_0^2 s_2^2 - \frac{95}{11} s_0^3 s_2 + s_0^4 \right) r^2 - 4/11 \zeta (s_0^4 - 8s_0^3 s_2 + 10s_2^3 s_0 - 4s_2^4) \right) r^8 R^4 \\
& + \frac{1}{8} r^{10} (s_0 - s_2) s_0 ((s_0^3 + 77s_0 s_2^2 - 36s_2^3 - 26s_0^2 s_2) r^2 + 24 (-2/3 s_2 + s_0) s_2^2 \zeta) R^2 - 1/8 s_2 ((s_0 - 9s_2) r^2 - 4\zeta s_2) r^{12} \\
& (s_0 - s_2)^2 s_0^2) R^4 \zeta^2 X(r) - 32 (R^4 - r^2 (s_0 - 2s_2) R^2 + r^4 s_0 s_2)^2 (R^2 + r^2 (s_0 - s_2))^2 (R^4 + 3 (-2/3 s_2 + s_0) r^2 R^2 \\
& + r^4 s_0 (s_0 - s_2)) \zeta^3 X_1(r) + 2 \left(-7 \left(\left(r^4 + \frac{64}{7} \zeta r^2 + \frac{48}{7} \zeta^2 \right) R^8 - \frac{8}{7} \left(\left(-\frac{15}{8} s_2 + s_0 \right) r^4 - 2\zeta (s_0 + 3s_2) r^2 - 6\zeta^2 s_0 \right) r^2 R^6 \right. \right. \\
& + \frac{4}{7} r^4 ((s_0 - 13/2 s_2) (s_0 - s_2) r^4 + \zeta (-11s_0^2 + 42s_0 s_2 - 19s_2^2) r^2 - 4\zeta^2 (-7s_0 s_2 + s_0^2 + 4s_2^2)) R^4 + \frac{3}{7} \left(\left(s_0^3 + \frac{71}{3} s_0 s_2^2 \right. \right. \\
& - 8s_2^3 - 14s_0^2 s_2) r^4 + 4 (-2s_2^2 + s_0^2 + 5/3 s_0 s_2) s_2 \zeta r^2 + \frac{32}{3} s_0 \zeta^2 s_2 (s_0 - s_2)) r^6 R^2 - 4/7 r^{10} s_0 s_2 (s_0 - s_2) (s_0 - 3s_2) \\
& (\zeta + r^2)) \zeta X_2(r) + \left(\left((16\zeta^3 + 32\zeta^2 r^2 + 6\zeta r^4) R^4 + \left((s_0 - 1/2 s_2) r^4 - 8 \left(s_0 - \frac{7}{8} s_2 \right) \zeta r^2 - 4\zeta^2 (s_0 - s_2) \right) r^4 R^2 + \right. \right. \\
& \left. \left. 1/2 r^8 s_0 (-2\zeta + r^2) (s_0 - s_2) \right) X_3(r) + R^4 X_4(r) \zeta r^4 \right) R^4) R^8 \Big\}, \tag{4}
\end{aligned}$$

where $X_i(r)$ and $i = 1 \cdots 4$ are defined as

$$\begin{aligned}
X(r) &= e \left[-2 \{ R^4 + 3(-2/3s_2 + s_0)r^2 R^2 + r^4 s_0(s_0 - s_2) \} \zeta e^{-\frac{s_2 r^2}{R^2}} + 2\zeta R^4 - 3s_2 r^4 R^2 \right] R^{-4} r^{-2}, \\
X_1(r) &= e \left[-2 \{ R^4 + 3(-2/3s_2 + s_0)r^2 R^2 + r^4 s_0(s_0 - s_2) \} \zeta e^{-\frac{s_2 r^2}{R^2}} + 2\zeta R^4 - 4s_2 r^4 R^2 \right] R^{-4} r^{-2}, \\
X_2(r) &= e \left[-2 \{ R^4 + 3(-2/3s_2 + s_0)r^2 R^2 + r^4 s_0(s_0 - s_2) \} \zeta e^{-\frac{s_2 r^2}{R^2}} + 2\zeta R^4 - 2s_2 r^4 R^2 \right] R^{-4} r^{-2}, \\
X_3(r) &= e \left[-2 \{ R^4 + 3(-2/3s_2 + s_0)r^2 R^2 + r^4 s_0(s_0 - s_2) \} \zeta e^{-\frac{s_2 r^2}{R^2}} + 2\zeta R^4 - s_2 r^4 R^2 \right] R^{-4} r^{-2}, \\
X_4(r) &= e^{-2 \left[\{ R^4 + 3(-2/3s_2 + s_0)r^2 R^2 + r^4 s_0(s_0 - s_2) \} e^{-\frac{s_2 r^2}{R^2}} - R^4 \right]} \zeta R^{-4} r^{-2}.
\end{aligned} \tag{5}$$

It has been shown that the KB ansatz relates the pressures and the density which effectively induces the EoSs as given by Eqs. (19) and (20). The coefficients in those equations are related to the model parameters as listed as below.

$$\begin{aligned}
a_1 &= - \{ (-23616\zeta_1^3 s_2^3 s_0^2 + 16704\zeta_1^3 s_2^2 s_0^3 - 5184\zeta_1^3 s_2 s_0^4 + 15120\zeta_1^3 s_2^4 s_0 + 576\zeta_1^3 s_0^5 - 3600\zeta_1^3 s_2^5 - 9252\zeta_1^2 s_0^2 s_2^2 \\
&\quad + 8988\zeta_1^2 s_0 s_2^3 + 3384\zeta_1^2 s_0^3 s_2 - 2832\zeta_1^2 s_2^4 - 360\zeta_1^2 s_0^4 + 24\zeta_1 s_0^3 - 362\zeta_1 s_2^3 + 822\zeta_1 s_0 s_2^2 - 408\zeta_1 s_0^2 s_2 + 12s_0 s_2 \\
&\quad - 3s_2^2) \} \{ -59040\zeta_1^3 s_2^3 s_0^2 + 41760\zeta_1^3 s_2^2 s_0^3 - 12960\zeta_1^3 s_2 s_0^4 + 37800\zeta_1^3 s_2^4 s_0 + 1440\zeta_1^3 s_0^5 - 9000\zeta_1^3 s_2^5 - 7620\zeta_1^2 s_2^4 \\
&\quad - 504\zeta_1^2 s_0^4 + 22884\zeta_1^2 s_0 s_2^3 + 6696\zeta_1^2 s_0^3 s_2 - 21636\zeta_1^2 s_0^2 s_2^2 - 72\zeta_1 s_0^3 + 1878\zeta_1 s_0 s_2^2 - 1130\zeta_1 s_2^3 - 384\zeta_1 s_0^2 s_2 \\
&\quad - 15s_2^2 \}^{-1}, \\
a_2 &= 2 \{ (-19642\zeta_1^2 s_2^4 s_0 + 4080\zeta_1^2 s_2 s_0^4 + 27112\zeta_1^2 s_2^3 s_0^2 - 16116\zeta_1^2 s_2^2 s_0^3 - 3414528\zeta_1^5 s_2^3 s_0^5 + 6113664\zeta_1^5 s_2^4 s_0^4 \\
&\quad + 4207680\zeta_1^5 s_2^6 s_0^2 - 1468800\zeta_1^5 s_2^7 s_0 + 1112832\zeta_1^5 s_2^2 s_0^6 - 193536\zeta_1^5 s_0^7 s_2 - 1232352\zeta_1^4 s_0 s_2^6 - 3180672\zeta_1^4 s_0^3 s_2^4 \\
&\quad - 632448\zeta_1^4 s_0^5 s_2^2 + 2768832\zeta_1^4 s_0^2 s_2^5 + 94464\zeta_1^4 s_0^6 s_2 + 1966464\zeta_1^4 s_0^4 s_2^3 - 251040\zeta_1^3 s_2^5 s_0 + 420024\zeta_1^3 s_2^4 s_0^2 \\
&\quad + 110448\zeta_1^3 s_2^2 s_0^4 - 320784\zeta_1^3 s_2^3 s_0^3 - 15264\zeta_1^3 s_0^5 s_2 + 1056\zeta_1 s_0^2 s_2^2 - 704\zeta_1 s_0 s_2^3 - 276\zeta_1 s_0^3 s_2 + 3s_2^3 + 3s_0 s_2^2 \\
&\quad - 72\zeta_1^2 s_0^5 + 5152\zeta_1^2 s_2^5 + 216000\zeta_1^5 s_2^8 + 13824\zeta_1^5 s_0^8 - 4608\zeta_1^4 s_0^7 + 220320\zeta_1^4 s_2^7 + 56040\zeta_1^3 s_2^6 + 288\zeta_1^3 s_0^6 - 72\zeta_1 s_0^4 \\
&\quad + 142\zeta_1 s_2^4 - 6587136\zeta_1^5 s_2^5 s_0^3) e^{-6(-s_2+s_0)\zeta_1} \} \{ (-59040\zeta_1^3 s_2^3 s_0^2 + 41760\zeta_1^3 s_2^2 s_0^3 - 12960\zeta_1^3 s_2 s_0^4 + 37800\zeta_1^3 s_2^4 s_0 \\
&\quad + 1440\zeta_1^3 s_0^5 - 9000\zeta_1^3 s_2^5 - 7620\zeta_1^2 s_2^4 - 504\zeta_1^2 s_0^4 + 22884\zeta_1^2 s_0 s_2^3 + 6696\zeta_1^2 s_0^3 s_2 - 21636\zeta_1^2 s_0^2 s_2^2 - 72\zeta_1 s_0^3 \\
&\quad + 1878\zeta_1 s_0 s_2^2 - 1130\zeta_1 s_2^3 - 384\zeta_1 s_0^2 s_2 - 15s_2^2) R^2 \kappa^2 \}^{-1}, \\
a_3 &= -2 \{ -23616\zeta_1^3 s_0^2 s_2^3 + 16704\zeta_1^3 s_0^3 s_2^2 + 15120\zeta_1^3 s_0 s_2^4 - 5184\zeta_1^3 s_0^4 s_2 - 3600\zeta_1^3 s_2^5 + 576\zeta_1^3 s_0^5 - 2832\zeta_1^2 s_2^4 \\
&\quad + 8898\zeta_1^2 s_2^3 s_0 - 9018\zeta_1^2 s_2^2 s_0^2 + 3204\zeta_1^2 s_0^3 s_2 - 324\zeta_1^2 s_0^4 + 30\zeta_1 s_0^3 - 362\zeta_1 s_2^3 - 360\zeta_1 s_2 s_0^2 + 774\zeta_1 s_2^2 s_0 - 3s_2^2 \\
&\quad + 9s_0 s_2 - 3s_0^2 \} \{ -72\zeta_1 s_0^3 + 6696\zeta_1^2 s_0^3 s_2 + 1878\zeta_1 s_2^2 s_0 - 384\zeta_1 s_2 s_0^2 - 1130\zeta_1 s_2^3 - 7620\zeta_1^2 s_2^4 - 504\zeta_1^2 s_0^4 \\
&\quad + 37800\zeta_1^3 s_0 s_2^4 - 59040\zeta_1^3 s_0^2 s_2^3 + 41760\zeta_1^3 s_0^3 s_2^2 - 12960\zeta_1^3 s_0^4 s_2 - 15s_2^2 + 22884\zeta_1^2 s_2^3 s_0 - 21636\zeta_1^2 s_2^2 s_0^2 \\
&\quad + 1440\zeta_1^3 s_0^5 - 9000\zeta_1^3 s_2^5 \}^{-1}, \\
a_4 &= - \{ (-504\zeta_1^2 s_0^5 - 6587136\zeta_1^5 s_2^5 s_0^3 - 3414528\zeta_1^5 s_2^3 s_0^5 + 6113664\zeta_1^5 s_2^4 s_0^4 - 1468800\zeta_1^5 s_2^7 s_0 + 4207680\zeta_1^5 s_2^6 s_0^2 \\
&\quad + 1112832\zeta_1^5 s_2^2 s_0^6 - 193536\zeta_1^5 s_0^7 s_2 + 3s_2^3 - 767904\zeta_1^3 s_2^3 s_0^3 - 330\zeta_1 s_2^2 s_0^2 + 54616\zeta_1^2 s_2^3 s_0^2 - 47232\zeta_1^3 s_0^5 s_2 \\
&\quad + 281952\zeta_1^3 s_2^2 s_0^4 - 60\zeta_1 s_0^3 s_2 - 592752\zeta_1^3 s_2^5 s_0 + 988104\zeta_1^3 s_2^4 s_0^2 + 982\zeta_1 s_2^4 - 1364\zeta_1 s_2^3 s_0 - 12876\zeta_1^2 s_2^2 s_0^3 \\
&\quad - 24\zeta_1^2 s_2 s_0^4 + 3086208\zeta_1^4 s_0^4 s_2^3 + 208512\zeta_1^4 s_0^6 s_2 - 1130112\zeta_1^4 s_0^5 s_2^2 + 3803040\zeta_1^4 s_0^2 s_2^5 - 4603680\zeta_1^4 s_0^3 s_2^4 \\
&\quad - 1634112\zeta_1^4 s_0 s_2^6 + 285120\zeta_1^4 s_2^7 - 14976\zeta_1^4 s_0^7 + 18s_0^2 s_2 - 24s_0 s_2^2 + 180\zeta_1 s_0^4 + 20452\zeta_1^2 s_2^5 + 133800\zeta_1^3 s_2^6 \\
&\quad + 3744\zeta_1^3 s_0^6 - 61420\zeta_1^2 s_2^4 s_0 + 216000\zeta_1^5 s_2^8 + 13824\zeta_1^5 s_0^8) e^{-6(-s_2+s_0)\zeta_1} \} \{ R^2 (-72\zeta_1 s_0^3 + 6696\zeta_1^2 s_0^3 s_2 + 1878\zeta_1 s_2^2 s_0 \\
&\quad - 384\zeta_1 s_2 s_0^2 - 1130\zeta_1 s_2^3 - 7620\zeta_1^2 s_2^4 - 504\zeta_1^2 s_0^4 + 37800\zeta_1^3 s_0 s_2^4 - 59040\zeta_1^3 s_0^2 s_2^3 + 41760\zeta_1^3 s_0^3 s_2^2 - 12960\zeta_1^3 s_0^4 s_2 \\
&\quad - 15s_2^2 + 22884\zeta_1^2 s_2^3 s_0 - 21636\zeta_1^2 s_2^2 s_0^2 + 1440\zeta_1^3 s_0^5 - 9000\zeta_1^3 s_2^5) \kappa^2 \}^{-1}.
\end{aligned} \tag{6}$$

Using the above set of equations, one can find the physical quantities appear in Eq. (19) and (20) in terms of the model parameters, where $v_r^2 = c_1$, $\rho_1 = \rho_s = -c_2/c_1$, $v_t^2 = c_3$ and $\rho_2 = -c_4/c_3$.

III. THE DENSITY AND PRESSURES GRADIENTS

Recalling the matter density and the pressures as obtained for the quadratic polynomial $f(R)$ gravity, namely Eqs. (1), (2), and (3) we obtain the gradients of these quantities with respect to the radial distance as below

$$\begin{aligned}
\rho' = \frac{1}{r^{11} R^{14} c^2 \kappa^2} \Bigg\{ & -512R^2 (R^4 - r^2 (s_0 - 2s_2) R^2 + r^4 s_0 s_2) \left[\left(R^6 + \frac{9}{4} r^2 \left(s_0 - \frac{5}{9} s_2 \right) R^4 + \frac{1}{2} r^4 \left(s_0^2 - s_2^2 + \frac{1}{2} s_0 s_2 \right) R^2 \right. \right. \\
& + 1/4 r^6 s_0 s_2 (s_0 - s_2) \Big) R^2 (R^4 - r^2 (s_0 - 2s_2) R^2 + r^4 s_0 s_2) (R^2 + r^2 (-s_2 + s_0)) \zeta_1 + \frac{15}{16} r^2 (R^{12} + 7/5 r^2 (s_0 + 4/7 s_2) R^{10} \\
& - \frac{4}{15} r^4 \left(\frac{25}{4} s_2^2 + s_0^2 - \frac{67}{4} s_0 s_2 \right) R^8 + r^6 \left(s_0^3 - \frac{16}{3} s_2 s_0^2 - \frac{88}{15} s_2^3 + \frac{182}{15} s_0 s_2^2 \right) R^6 + \frac{13}{15} r^8 \left(\frac{68}{13} s_2^4 - \frac{320}{13} s_2^3 s_0 - \frac{149}{13} s_0^3 s_2 \right. \\
& + s_0^4 + \frac{405}{13} s_2^2 s_0^2 \Big) R^4 - \frac{12}{5} (-s_2 + s_0) r^{10} s_2 \left(\frac{17}{9} s_2^2 - \frac{23}{6} s_0 s_2 + s_0^2 \right) s_0 R^2 + \frac{17}{15} r^{12} s_0^2 s_2^2 (-s_2 + s_0)^2 \Big] X_5 \zeta_1^3 (R^2 \\
& + r^2 (s_0 - s_2)) + 768R^4 \zeta_1^2 (R^6 (R^4 - r^2 (s_0 - 2s_2) R^2 + r^4 s_0 s_2) (R^6 + 3/2 r^2 (s_0 - 1/3 s_2) R^4 + 3/2 r^4 (-2/3 s_2 + s_0) s_2 R^2 \\
& + \frac{r^6}{2} s_0 s_2 (s_0 - s_2)) (R^2 + r^2 (s_0 - s_2)) \zeta_1^2 + \frac{15}{8} r^2 R^2 \left(R^{14} + \frac{14}{15} r^2 \left(s_0 + \frac{19}{14} s_2 \right) R^{12} - \frac{38}{45} r^4 \left(s_0^2 - \frac{127}{19} s_0 s_2 + \frac{89}{38} s_2^2 \right) R^{10} \right. \\
& - \frac{2r^6}{15} \left(s_0^3 + \frac{155}{6} s_2^3 - \frac{122}{3} s_0 s_2^2 - \frac{7}{3} s_2 s_0^2 \right) R^8 + \frac{17}{45} r^8 \left(\frac{22}{17} s_2^2 - \frac{88}{17} s_0 s_2 + s_0^2 \right) (4s_2^2 - 6s_0 s_2 + s_0^2) R^6 - \frac{13}{15} r^{10} \\
& \times \left(\frac{50}{39} s_2^3 s_0 - \frac{12}{13} s_2^4 - \frac{64}{13} s_0^3 s_2 + s_0^4 + \frac{10}{3} s_2^2 s_0^2 \right) s_2 R^4 + \frac{7}{45} (s_0 - s_2) r^{12} \left(-\frac{36}{7} s_2^2 + \frac{47}{7} s_0 s_2 + s_0^2 \right) s_2^2 s_0 R^2 \\
& + 1/5 r^{14} s_0^2 s_2^3 (-s_2 + s_0)^2 \Big) \zeta_1 + \frac{23}{48} r^4 \left(R^{14} + \frac{9}{46} r^2 \left(\frac{34}{3} s_2 + s_0 \right) R^{12} - \frac{9}{23} r^4 \left(s_0^2 - \frac{41}{9} s_2^2 - 4s_0 s_2 \right) R^{10} - \frac{3}{46} r^6 \left(\frac{86}{3} s_2 s_0^2 \right. \right. \\
& + s_0^3 - \frac{305}{3} s_0 s_2^2 + \frac{82}{3} s_2^3 \Big) R^8 - \frac{21}{46} r^8 \left(\frac{152}{7} s_2^4 - \frac{90}{7} s_0^3 s_2 + s_0^4 - \frac{1228}{21} s_2^3 s_0 + \frac{940}{21} s_2^2 s_0^2 \right) R^6 + \frac{3}{23} r^{10} (367 s_0^2 s_2^3 \\
& + 44 s_2^5 - \frac{363}{2} s_0^3 s_2^2 + s_0^5 - 242 s_2^4 s_0 + \frac{45}{2} s_0^4 s_2 \Big) R^4 + 1/46 r^{12} s_0 (-s_2 + s_0) (-264 s_2^4 + 610 s_2^3 s_0 - 204 s_2^2 s_0^2 + s_0^4 \\
& - 11 s_0^3 s_2) R^2 - 1/46 r^{14} s_0^2 s_2 (s_0^2 - s_0 s_2 - 66 s_2^2) (-s_2 + s_0)^2 \Big) X_5 + 128 (R^4 - r^2 (s_0 - 2s_2) R^2 + r^4 s_0 s_2)^3 (R^4 + 3r^2 (s_0 \\
& - 2/3 s_2 +) R^2 + r^4 s_0 (-s_2 + s_0)) \zeta_1^4 (R^2 + r^2 (-s_2 + s_0))^3 X_6 - 512R^6 ((R^{10} (R^6 + 3/4 r^2 (s_0 + 1/3 s_2) R^4 - 1/2 r^4 (3s_2^2 \\
& + s_0^2 - \frac{11}{2} s_0 s_2) R^2 + \frac{3}{4} r^6 s_0 s_2 (s_0 - s_2)) \zeta_1^3 + \frac{45}{16} r^2 R^6 \left(R^8 + \frac{7}{15} r^2 \left(s_0 + \frac{11}{7} s_2 \right) R^6 - \frac{19}{45} r^4 \left(s_0^2 - \frac{106}{19} s_0 s_2 + \frac{46}{19} s_2^2 \right) R^4 \right. \\
& + \frac{11}{45} r^6 \left(s_0 - \frac{8}{11} s_2 \right) s_2 (3s_2 + s_0) R^2 + \frac{4}{15} r^8 s_0 s_2^2 (-s_2 + s_0)) \zeta_1^2 + \frac{23}{16} r^4 R^2 \left(R^{10} + \frac{3}{46} r^2 \left(s_0 + \frac{62}{3} s_2 \right) R^8 - \frac{19}{92} r^4 (s_0^2 \\
& - \frac{61}{19} s_0 s_2 - \frac{50}{19} s_2^2) R^6 - \frac{3}{92} r^6 \left(23 s_2 s_0^2 + \frac{100}{3} s_2^3 + s_0^3 - 79 s_0 s_2^2 \right) R^4 - \frac{3}{46} r^8 s_2 \left(-\frac{19}{3} s_2 s_0^2 + 4s_2^3 - 2/3 s_0 s_2^2 + s_0^3 \right) R^2 \\
& - \frac{1}{92} r^{10} s_0 s_2 (-s_2 + s_0) (3s_2 + s_0) (-4s_2 + s_0) \Big) \zeta_1 + \frac{9}{128} r^6 (R^{10} - 1/3 r^2 (s_0 - 5s_2) R^8 - 1/9 r^4 (2s_0 s_2 - 12s_2^2 + s_0^2) R^6 \\
& + 2/3 r^6 \left(\frac{31}{6} s_2 s_0^2 + \frac{50}{3} s_2^3 + s_0^3 - \frac{157}{6} s_0 s_2^2 \right) R^4 + 2/9 r^8 (s_0^4 - 24s_2^4 - 8s_0^3 s_2 + 79s_2^3 s_0 - 36s_2^2 s_0^2) R^2 - 2/9 r^{10} s_0 s_2 (s_0 \\
& - s_2) (3s_2 + s_0) (-4s_2 + s_0)) \zeta_1 X_7 - \frac{1}{4} R^4 \left(\left(R^{12} \zeta_1^4 + \left(\frac{3}{4} R^8 r^4 s_2 + \frac{15}{4} r^2 R^{10} \right) \zeta_1^3 + \frac{23}{8} r^4 R^4 \left(R^4 - \frac{3}{46} r^2 \left(s_0 - \frac{22}{3} s_2 \right) R^2 \right. \right. \right. \\
& - 1/46 r^4 (s_0 + s_2) (s_0 - 2s_2)) \zeta_1^2 + \frac{5}{16} r^6 R^2 (R^4 - 3/10 r^2 (s_0 - 7/3 s_2) R^2 - 1/10 r^4 (s_0 + s_2) (s_0 - 2s_2)) \zeta_1 + \frac{1}{64} r^8 (R^2 \\
& + 2r^2 s_2) (R^2 - r^2 s_2)) X_8 - 1/32 r^6 (\zeta_1 R^2 + 1/2 r^2) R^4 X_9) \Bigg\}, \tag{7}
\end{aligned}$$

$$\begin{aligned}
p'_r = & \frac{1}{R^{12}r^9\kappa^2} \left(192R^2 \left(\zeta_1 R^{16} + \frac{5}{2}r^2 \left(\zeta_1 s_0 + \frac{1}{3} \right) R^{14} + \frac{r^4}{3} \left((20s_0s_2 - 11s_2^2 + s_0^2) \zeta_1 + \frac{9}{4}s_2 + \frac{33}{8}s_0 \right) R^{12} - \frac{7}{3}r^6 \left(\left(\frac{9}{2}s_0s_2^2 \right. \right. \right. \\
& - 6s_2s_0^2 + s_0^3 - \frac{4}{7}s_2^3 \Big) \zeta_1 + \frac{3}{28}s_0^2 - \frac{16}{7}s_0s_2 + \frac{25}{28}s_2^2 \Big) R^{10} - \frac{4}{3}r^8 \left(\left(\frac{13}{4}s_2^2s_0^2 - 6s_0^3s_2 + 2s_2^3s_0 + s_0^4 - s_2^4 \right) \zeta_1 + 9/4s_2^3 \right. \\
& - \frac{13}{8}s_2s_0^2 - \frac{87}{32}s_0s_2^2 + \frac{5}{32}s_0^3 \Big) R^8 - 1/6r^{10} \left(s_0 (s_0^4 + 30s_2^3s_0 - 8s_0^3s_2 - 12s_2^2s_0^2 - 12s_2^4) \zeta_1 - \frac{15}{4}s_0^4 - \frac{223}{2}s_2^2s_0^2 \right. \\
& - 18s_2^4 + 90s_2^3s_0 + 34s_0^3s_2 \Big) R^6 + 1/6s_0r^{12} \left(8s_0 (s_0 - 3/4s_2) s_2^2 (-s_2 + s_0) \zeta_1 + s_0^4 - \frac{89}{4}s_0^3s_2 + \frac{361}{4}s_2^2s_0^2 + 27s_2^4 \right. \\
& - \frac{187}{2}s_2^3s_0 \Big) R^4 - 1/24s_0^2 (-4s_0s_2^2 (-s_2 + s_0) \zeta_1 + s_0^3 + 54s_2^3 + 11s_2s_0^2 - 86s_0s_2^2) r^{14} (-s_2 + s_0) R^2 + 1/24r^{16}s_0^3s_2 (s_0 \\
& + 9s_2) (-s_2 + s_0)^2 \Big) \zeta_1^2 X_{10} - 192R^4 \zeta_1 \left(\zeta_1^2 R^{14} + 3/2 \left(\zeta_1 s_0 + \frac{10}{9} \right) r^2 \zeta_1 R^{12} + 1/6r^4 \left(\frac{15}{8} + (14s_0s_2 + s_0^2 - 8s_2^2) \zeta_1^2 + (5s_2 (s_0 \right. \right. \\
& + 21/2s_0) \zeta_1) R^{10} - \frac{1}{6} \left(s_0 (8s_2^2 - 11s_0s_2 + s_0^2) \zeta_1^2 + \left(\frac{23}{2}s_2^2 - \frac{105}{4}s_0s_2 + 5/4s_0^2 \right) \zeta_1 - \frac{21}{8}s_2 - \frac{3}{8}s_0 \right) r^6 R^8 + \frac{1}{3}r^8 (s_0^2s_2 \\
& (-s_2 + s_0) \zeta_1^2 + \left(-5/8s_0^3 - \frac{19}{8}s_0s_2^2 + \frac{59}{8}s_2s_0^2 - 3/2s_2^3 \right) \zeta_1 + 1/16s_0^2 + \frac{7}{8}s_2^2 + 1/4s_0s_2 \Big) R^6 + \frac{5}{12}r^{10} (s_0s_2 (3/5s_0s_2 \\
& - 6/5s_2^2 + s_0^2) \zeta_1 - \frac{33}{20}s_2s_0^2 - 6/5s_2^3 + \frac{71}{20}s_0s_2^2 + 1/10s_0^3 \Big) R^4 + 1/24s_0 (s_0s_2 (3s_2 + s_0) (-s_2 + s_0) \zeta_1 - s_0^3 - 12s_2^3 \\
& - 3s_2s_0^2 + 20s_0s_2^2) r^{12} R^2 + 1/24r^{14}s_0^2s_2 (3s_2 + s_0) (-s_2 + s_0) X_7 - 64\zeta_1^3 (R^4 + 3r^2 (-2/3s_2 + s_0) R^2 + r^4s_0 (-s_2 + s_0)) \\
& (R^4 - r^2 (s_0 - 2s_2) R^2 + r^4s_0s_2)^2 (R^2 + 1/2s_0r^2) (R^2 + r^2 (-s_2 + s_0))^2 X_5 + 64R^8 ((\zeta_1^3 R^{10} + 1/2r^2 \zeta_1^2 (\zeta_1 s_0 + 5) R^8 \\
& + \frac{9}{8}r^4 \zeta_1 \left(\frac{7}{9} + \left(\frac{2}{9}s_2 + s_0 \right) \zeta_1 \right) R^6 + \frac{1}{8} \left(s_0 (s_0 + s_2) \zeta_1^2 - \frac{1}{4} + \left(\frac{3}{2}s_0 + 5/2s_2 \right) \zeta_1 \right) r^6 R^4 + \frac{r^8}{16} \left(s_0 (s_0 + s_2) \zeta_1 - \frac{1}{2}s_2 \right) R^2 \\
& - 1/16r^{10}s_0s_2) X_8 + 1/16R^4 X_9 r^4 (\zeta_1 R^2 + 1/2r^2)) , \tag{8}
\end{aligned}$$

$$\begin{aligned}
X_5 &= e \left[-2\zeta_1 \{ R^4 + 3r^2(-2/3s_2 + s_0)R^2 + r^4s_0(-s_2 + s_0) \} e^{-\frac{r^2s_2}{R^2}} + 2\zeta_1 R^4 - 4s_2 r^4 \right] R^{-2} r^{-2}, \\
X_6 &= e \left[-2\zeta_1 \{ R^4 + 3r^2(-2/3s_2 + s_0)R^2 + r^4s_0(-s_2 + s_0) \} e^{-\frac{r^2s_2}{R^2}} + 2\zeta_1 R^4 - 5s_2 r^4 \right] R^{-2} r^{-2}, \\
X_7 &= e \left[-2\zeta_1 \{ R^4 + 3r^2(-2/3s_2 + s_0)R^2 + r^4s_0(-s_2 + s_0) \} e^{-\frac{r^2s_2}{R^2}} + 2\zeta_1 R^4 - 2s_2 r^4 \right] R^{-2} r^{-2}, \\
X_8 &= e \left[-2\zeta_1 (R^4 + 3r^2(-2/3s_2 + s_0)R^2 + r^4s_0(-s_2 + s_0)) e^{-\frac{r^2s_2}{R^2}} + 2\zeta_1 R^4 - s_2 r^4 \right] R^{-2} r^{-2}, \\
X_9 &= e^{-2} \left[(R^4 + 3r^2(-2/3s_2 + s_0)R^2 + r^4s_0(-s_2 + s_0)) e^{-\frac{r^2s_2}{R^2}} - R^4 \right] \zeta_1 R^{-2} r^{-2}, \\
X_{10} &= e \left[-2\zeta_1 \{ R^4 + 3r^2(-2/3s_2 + s_0)R^2 + r^4s_0(-s_2 + s_0) \} e^{-\frac{r^2s_2}{R^2}} + 2\zeta_1 R^4 - 3s_2 r^4 \right] R^{-2} r^{-2}
\end{aligned}$$

(10)