ON SOME TRACE INEQUALITIES FOR POSITIVE DEFINITE HERMITIAN MATRICES

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Abstract. Let $\boldsymbol{A}$ be a positive definite Hermitian matrix, we investigate the trace inequalities of $\boldsymbol{A}$. By using the equivalence of the deformed matrix, according to some properties of positive definite Hermitian matrices and some elementary inequalities, we extend some previous works about the trace inequalities for positive definite Hermitian matrices, and obtain some valuable theory.

1. Introduction

In mathematics, a Hermitian matrix (or self-adjoint matrix) is a square matrix with complex entries that is equal to its own conjugate transpose. That is, the elements in the i-th row and j-th column is equal to the complex conjugate of the elements in the j-th row and i-th column. In other words, the matrix $\boldsymbol{A}$ is Hermitian if and only if $\boldsymbol{A} = \overline{\boldsymbol{A}}^T$, where $\overline{\boldsymbol{A}}^T$ denotes the conjugate transpose of matrix $\boldsymbol{A}$.

Hermitian matrices play an important role in statistical mechanics[1], engineering, such as communication, a n-dimension signals cross-correlation properties, just as conjugate symmetry, then we can use Hermitian matrices to describe.

The earliest study of matrix inequality works in the literature [2]. 1980, R. Bellman[3] prove some trace inequalities for positive definite Hermitian matrices.

(1) $tr(\boldsymbol{AB})^2 \leq tr(\boldsymbol{A}^2\boldsymbol{B}^2)$;
(2) $2tr(\boldsymbol{AB}) \leq tr\boldsymbol{A}^2 + tr\boldsymbol{B}^2$;
(3) $tr(\boldsymbol{AB}) \leq (tr\boldsymbol{A}^2)^{\frac{1}{2}}(tr\boldsymbol{B}^2)^{\frac{1}{2}}$.

Since then, the problems of trace inequality for positive definite(semidefinite) Hermitian matrices have caused the attention of scholars, get a lot of interesting results. There exists a vast literature that studies the trace(see [4],[5],[6],[7],[8]). In the paper, using identical deformation of matrix, and combined with some elementary inequality, our purpose is to derive some new results on trace inequality for positive definite Hermitian matrices.

The rest of this paper is organized as follows. In Section 2, we will give relevant definitions and properties of Hermitian matrices; In Section 3, we will quote some lemmas; In Section 4, which is the main part of the paper, using properties of Hermitian matrices, we investigate the trace inequalities for positive definite Hermitian matrices.

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2. Preliminaries

By $\boldsymbol{M}_{n,m}(\boldsymbol{F})$ we denote the n-by-m matrices over a field $\boldsymbol{F}$, usually the real numbers $\boldsymbol{R}$ or the complex numbers $\boldsymbol{C}$. Most often, the facts discussed are valid in the setting of the complex-entried matrices, in which case $\boldsymbol{M}_{n,m}(\boldsymbol{C})$ is abbreviated as $\boldsymbol{M}_{n,m}$. In case of square matrices we replaced $\boldsymbol{M}_{n,n}$ by $\boldsymbol{M}_n$.

Let $\boldsymbol{A} = (a_{ij}) \in \boldsymbol{M}_n$, we may denote the eigenvalues of $\boldsymbol{A}$ by $\lambda_1, \lambda_2, \ldots, \lambda_n$, without loss of generality, where let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let $\sigma(\boldsymbol{A})$ denote the singular value, and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. It is known that if $\boldsymbol{A}$ is Hermitian, then all eigenvalues of $\boldsymbol{A}$ are real numbers and if $\boldsymbol{A}$ is unitary, then every eigenvalue of $\boldsymbol{A}$ has modulus one. The sum of two Hermitian matrices of the same size is Hermitian. If $\boldsymbol{A}$ is Hermitian, then $\boldsymbol{A}^k$ is Hermitian for all $k = 1, 2, \ldots$; If $\boldsymbol{A}$ is invertible as well, then $\boldsymbol{A}^{-1}$ is Hermitian.

A Hermitian matrix $\boldsymbol{A} \in \boldsymbol{M}_n$ is said to be positive semidefinite, denoted by $\boldsymbol{A} \geq 0$, if $(\boldsymbol{A}x, x) \geq 0$ for all $x \in \boldsymbol{C}^n$, and it is called positive definite, denoted by $\boldsymbol{A} > 0$, if $(\boldsymbol{A}x, x) > 0$ for all nonzero $x \in \boldsymbol{C}^n$ ($\boldsymbol{C}^n$ denotes complex vector spaces), where (.) denotes the Euclidean inner product on $\boldsymbol{C}^n$.

Remark 2.1.
1. The sum of any two positive definite matrices of the same size is positive definite;
2. Each eigenvalue of a positive definite matrix is a nonnegative (positive) real number;
3. The trace and the determinant of a positive definite matrix are non-negative (positive) real numbers;
4. Any principal submatrix of a positive definite matrix is positive definite.

A Hermitian matrix is positive definite if and only if all of its eigenvalues are nonnegative (positive) real numbers.

We will use several times the fact.

Let $\boldsymbol{A} \in \boldsymbol{M}_n$. Then the trace of $\boldsymbol{A}$ is given by $tr\boldsymbol{A} = \sum_{i=1}^n a_{ii}$. The trace function has the following properties.

Let $\boldsymbol{A}, \boldsymbol{B} \in \boldsymbol{M}_n$, $\alpha \in \boldsymbol{C}$. Then
1. $tr(\boldsymbol{A} + \boldsymbol{B}) = tr\boldsymbol{A} + tr\boldsymbol{B}$.
2. $tr\alpha\boldsymbol{A} = \alpha tr\boldsymbol{A}$.
3. $tr(\boldsymbol{AB}) = tr(\boldsymbol{BA})$.

3. Some Lemmas

The following lemmas play a fundamental role in this paper.

Lemma 3.1. [9] *Let* $\mathbf{A}_i \in \mathbf{M}_n (i = 1, 2, \ldots, m)$. *Then*

$$|tr(\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_m)| \leq \sum_{i=1}^n \sigma_i(\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_m) \leq \sum_{i=1}^n \sigma_i(\mathbf{A}_1)\sigma_i(\mathbf{A}_2) \cdots \sigma_i(\mathbf{A}_m),$$

where $\sigma_1(\mathbf{A}_i) \geq \sigma_2(\mathbf{A}_i) \geq \cdots \geq \sigma_n(\mathbf{A}_i)$.

Lemma 3.2. [10] *Let* $\alpha_i > 0 (i = 1, 2, \ldots, n)$, *and* $\sum_{i=1}^n \alpha_i \geq 1$. *Let* $a_{ij} > 0 (j = 1, 2, \ldots, m)$. *Then*

$$\sum_{j=1}^m a_{1j}^{\alpha_1} a_{2j}^{\alpha_2} \cdots a_{nj}^{\alpha_n} \leq (\sum_{j=1}^m a_{1j})^{\alpha_1} (\sum_{j=1}^m a_{2j})^{\alpha_2} \cdots (\sum_{j=1}^m a_{nj})^{\alpha_n}.$$

Lemma 3.3. [11] *Let $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ be same size positive definite matrices, and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are positive real numbers, and $\sum_{i=1}^{n} \alpha_i = 1$. Then*

$$tr \prod_{i=1}^{n} \mathbf{A}_i^{\alpha_i} \leq \sum_{i=1}^{n} \alpha_i tr \mathbf{A}_i$$

4. Main results

Now, we prove the following theorem.

Theorem 4.1. *Let $\mathbf{A}_i, \mathbf{B}_i (i = 1, 2, \ldots, m)$ be same size positive definite matrices, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$tr \sum_{i=1}^{m} (\mathbf{A}_i \mathbf{B}_i) \leq (\sum_{i=1}^{m} tr \mathbf{A}_i^p)^{\frac{1}{p}} (\sum_{i=1}^{m} tr \mathbf{B}_i^q)^{\frac{1}{q}}.$$

Proof. Since the eigenvalues and traces of positive definite matrices are all positive real numbers, the eigenvalues are equal to the singular values. Then, according Lemma 3.1 and spectral mapping theorem, we have

$$\begin{aligned}
&\sum_{i=1}^{m} tr \boldsymbol{A}_i^{\frac{1}{p}} \boldsymbol{B}_i^{\frac{1}{q}} \\
&\leq \sum_{i=1}^{m} \sum_{j=1}^{n} \sigma_j (\boldsymbol{A}_i^{\frac{1}{p}} \boldsymbol{B}_i^{\frac{1}{q}}) \\
&\leq \sum_{i=1}^{m} \sum_{j=1}^{n} \sigma_j (\boldsymbol{A}_i^{\frac{1}{p}}) \sigma_j (\boldsymbol{B}_i^{\frac{1}{q}}) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_j (\boldsymbol{A}_i^{\frac{1}{p}}) \lambda_j (\boldsymbol{B}_i^{\frac{1}{q}}) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_j^{\frac{1}{p}} (\boldsymbol{A}_i) \lambda_j^{\frac{1}{q}} (\boldsymbol{B}_i).
\end{aligned} \tag{4.1}$$

By using Lemma 3.2, we get

$$\begin{aligned}
&\sum_{i=1}^{m} tr \boldsymbol{A}_i^{\frac{1}{p}} \boldsymbol{B}_i^{\frac{1}{q}} \\
&\leq \sum_{i=1}^{m} (\sum_{j=1}^{n} \lambda_j (\boldsymbol{A}_i))^{\frac{1}{p}} (\sum_{j=1}^{n} \lambda_j (\boldsymbol{B}_i))^{\frac{1}{q}} \\
&\leq \sum_{i=1}^{m} (tr \boldsymbol{A}_i)^{\frac{1}{p}} (tr \boldsymbol{B}_i)^{\frac{1}{q}} \\
&\leq (\sum_{i=1}^{m} tr \boldsymbol{A}_i)^{\frac{1}{p}} (\sum_{i=1}^{m} tr \boldsymbol{B}_i)^{\frac{1}{q}}.
\end{aligned} \tag{4.2}$$

Let $\boldsymbol{A}_i = \boldsymbol{A}_i^p, \boldsymbol{B}_i = \boldsymbol{B}_i^q$. Then, we obtain

$$\sum_{i=1}^{m} tr (\boldsymbol{A}_i \boldsymbol{B}_i) \leq (\sum_{i=1}^{m} tr \boldsymbol{A}_i^p)^{\frac{1}{p}} (\sum_{i=1}^{m} tr \boldsymbol{B}_i^q)^{\frac{1}{q}}.$$

By (2.1), that is

$$tr\sum_{i=1}^{m}(\boldsymbol{A}_i\boldsymbol{B}_i) \le (\sum_{i=1}^{m} tr\boldsymbol{A}_i^p)^{\frac{1}{p}}(\sum_{i=1}^{m} tr\boldsymbol{B}_i^q)^{\frac{1}{q}}.$$

Thus, the proof is completed. □

Next, we give a trace inequality for positive definite matrices.

Theorem 4.2. *Let $\alpha_i > 0 (i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n}\alpha_i = 1$. $\mathbf{A}_i, \mathbf{B}_i, \mathbf{C}_i (i = 1, 2, \ldots, n)$ are same size positive definite matrices. Then*

$$tr[\prod_{i=1}^{n}\mathbf{A}_i^{\alpha_i} + \prod_{i=1}^{n}\mathbf{B}_i^{\alpha_i} + \prod_{i=1}^{n}\mathbf{C}_i^{\alpha_i}] \le \prod_{i=1}^{n}[tr(\mathbf{A}_i + \mathbf{B}_i + \mathbf{C}_i)]^{\alpha_i}.$$

Proof. Since trace of a matrix is a linear operation, by using Lemma 3.3, it follows that

(4.3)

$$\begin{aligned}
&\frac{tr(\prod_{i=1}^{n}\boldsymbol{A}_i^{\alpha_i} + \prod_{i=1}^{n}\boldsymbol{B}_i^{\alpha_i} + \prod_{i=1}^{n}\boldsymbol{C}_i^{\alpha_i})}{\prod_{i=1}^{n}[tr(\boldsymbol{A}_i + \boldsymbol{B}_i + \boldsymbol{C}_i)]^{\alpha_i}}\\
&= tr\prod_{i=1}^{n}[\frac{\boldsymbol{A}_i^{\alpha_i}}{tr(\boldsymbol{A}_i + \boldsymbol{B}_i + \boldsymbol{C}_i)^{\alpha_i}} + \frac{\boldsymbol{B}_i^{\alpha_i}}{tr(\boldsymbol{A}_i + \boldsymbol{B}_i + \boldsymbol{C}_i)^{\alpha_i}} + \frac{\boldsymbol{C}_i^{\alpha_i}}{tr(\boldsymbol{A}_i + \boldsymbol{B}_i + \boldsymbol{C}_i)^{\alpha_i}}]\\
&= tr\prod_{i=1}^{n}[\frac{\boldsymbol{A}_i}{tr(\boldsymbol{A}_i + \boldsymbol{B}_i + \boldsymbol{C}_i)}]^{\alpha_i} + tr\prod_{i=1}^{n}[\frac{\boldsymbol{B}_i}{tr(\boldsymbol{A}_i + \boldsymbol{B}_i + \boldsymbol{C}_i)}]^{\alpha_i} + tr\prod_{i=1}^{n}[\frac{\boldsymbol{C}_i}{tr(\boldsymbol{A}_i + B_i + \boldsymbol{C}_i)}]^{\alpha_i}\\
&\le \sum_{i=1}^{n}\frac{\alpha_i tr\boldsymbol{A}_i}{tr(\boldsymbol{A}_i + \boldsymbol{B}_i + \boldsymbol{C}_i)} + \sum_{i=1}^{n}\frac{\alpha_i tr\boldsymbol{B}_i}{tr(\boldsymbol{A}_i + \boldsymbol{B}_i + \boldsymbol{C}_i)} + \sum_{i=1}^{n}\frac{\alpha_i tr\boldsymbol{C}_i}{tr(\boldsymbol{A}_i + \boldsymbol{B}_i + \boldsymbol{C}_i)}\\
&= \sum_{i=1}^{n}\frac{\alpha_i(tr\boldsymbol{A}_i + tr\boldsymbol{B}_i + tr\boldsymbol{C}_i)}{tr(\boldsymbol{A}_i + \boldsymbol{B}_i + \boldsymbol{C}_i)}\\
&= \sum_{i=1}^{n}\alpha_i = 1.
\end{aligned}$$

Then, we have

$$tr[\prod_{i=1}^{n}\boldsymbol{A}_i^{\alpha_i} + \prod_{i=1}^{n}\boldsymbol{B}_i^{\alpha_i} + \prod_{i=1}^{n}\boldsymbol{C}_i^{\alpha_i}] \le \prod_{i=1}^{n}[tr(\boldsymbol{A}_i + \boldsymbol{B}_i + \boldsymbol{C}_i)]^{\alpha_i}.$$

□

Now we use mathematical induction to deduce our third result.

Theorem 4.3. *Let $\mathbf{A}_i (i = 1, 2, \ldots, n)$ be same size positive definite matrices. Then, we have the inequality*

$$\frac{tr(\sum_{i=1}^{n}\mathbf{A}_i)^2}{tr(\sum_{i=1}^{n}\mathbf{A}_i)} \le \sum_{i=1}^{n}\frac{tr\mathbf{A}_i^2}{tr\mathbf{A}_i}.$$

Proof. Since when $n=2$, according to (2) in the first page, we have

$$\begin{aligned}
&2tr\boldsymbol{A}_1\cdot tr\boldsymbol{A}_2\cdot tr(\boldsymbol{A}_1\boldsymbol{A}_2)\\
&\le 2tr\boldsymbol{A}_1\cdot tr\boldsymbol{A}_2\cdot(tr\boldsymbol{A}_1^2)^{\frac{1}{2}}(tr\boldsymbol{A}_2^2)^{\frac{1}{2}}\\
&=2[tr\boldsymbol{A}_1(tr\boldsymbol{A}_2^2)^{\frac{1}{2}}][tr\boldsymbol{A}_2(tr\boldsymbol{A}_1^2)^{\frac{1}{2}}]\\
&\le[tr\boldsymbol{A}_1(tr\boldsymbol{A}_2^2)^{\frac{1}{2}}]^2+[tr\boldsymbol{A}_2(tr\boldsymbol{A}_1^2)^{\frac{1}{2}}]^2\\
&=(tr\boldsymbol{A}_1)^2tr\boldsymbol{A}_2^2+(tr\boldsymbol{A}_2)^2tr\boldsymbol{A}_1^2.
\end{aligned}\tag{4.4}$$

that is

$$2tr(\boldsymbol{A}_1\boldsymbol{A}_2)\le\frac{tr\boldsymbol{A}_1\cdot tr\boldsymbol{A}_2^2}{tr\boldsymbol{A}_2}+\frac{tr\boldsymbol{A}_2\cdot tr\boldsymbol{A}_1^2}{tr\boldsymbol{A}_1}.\tag{4.5}$$

On the other hand,

$$tr(\boldsymbol{A}_1+\boldsymbol{A}_2)^2=tr\boldsymbol{A}_1^2+2tr(\boldsymbol{A}_1\boldsymbol{A}_2)+tr\boldsymbol{A}_2^2.\tag{4.6}$$

Note that

$$\begin{aligned}
&(\frac{tr\boldsymbol{A}_1^2}{tr\boldsymbol{A}_1}+\frac{tr\boldsymbol{A}_2^2}{tr\boldsymbol{A}_2})tr(\boldsymbol{A}_1+\boldsymbol{A}_2)\\
&=(\frac{tr\boldsymbol{A}_1^2}{tr\boldsymbol{A}_1}+\frac{tr\boldsymbol{A}_2^2}{tr\boldsymbol{A}_2})(tr\boldsymbol{A}_1+tr\boldsymbol{A}_2)\\
&=tr\boldsymbol{A}_1^2+tr\boldsymbol{A}_2^2+\frac{tr\boldsymbol{A}_1\cdot tr\boldsymbol{A}_2^2}{tr\boldsymbol{A}_2}+\frac{tr\boldsymbol{A}_2\cdot tr\boldsymbol{A}_1^2}{tr\boldsymbol{A}_1}.
\end{aligned}\tag{4.7}$$

So,according to (4.5), (4.6), we deduce

$$\begin{aligned}
&(\frac{tr\boldsymbol{A}_1^2}{tr\boldsymbol{A}_1}+\frac{tr\boldsymbol{A}_2^2}{tr\boldsymbol{A}_2})tr(\boldsymbol{A}_1+\boldsymbol{A}_2)\\
&\ge tr\boldsymbol{A}_1^2+2tr(\boldsymbol{A}_1\boldsymbol{A}_2)+tr\boldsymbol{A}_2^2=tr(\boldsymbol{A}_1+\boldsymbol{A}_2)^2.
\end{aligned}\tag{4.8}$$

Hence,

$$\frac{tr(\boldsymbol{A}_1+\boldsymbol{A}_2)^2}{tr(\boldsymbol{A}_1+\boldsymbol{A}_2)}\le\frac{tr\boldsymbol{A}_1^2}{tr\boldsymbol{A}_1}+\frac{tr\boldsymbol{A}_2^2}{tr\boldsymbol{A}_2}.\tag{4.9}$$

Then, the inequality holds.

Suppose that the inequality holds when $n=k$, i.e.,

$$\frac{tr(\sum_{i=1}^k\boldsymbol{A}_i)^2}{tr(\sum_{i=1}^k\boldsymbol{A}_i)}\le\sum_{i=1}^k\frac{tr\boldsymbol{A}_i^2}{tr\boldsymbol{A}_i}.$$

Then for $n=k+1$, we have

$$\begin{aligned}
&\frac{tr(\sum_{i=1}^{k+1}\boldsymbol{A}_i)^2}{tr(\sum_{i=1}^{k+1}\boldsymbol{A}_i)}=\frac{tr(\sum_{i=1}^k\boldsymbol{A}_i+\boldsymbol{A}_{k+1})^2}{tr(\sum_{i=1}^k\boldsymbol{A}_i+\boldsymbol{A}_{k+1})}\\
&\le\frac{tr(\sum_{i=1}^k\boldsymbol{A}_i)^2}{tr(\sum_{i=1}^k\boldsymbol{A}_i)}+\frac{tr\boldsymbol{A}_{k+1}^2}{tr\boldsymbol{A}_{k+1}}\\
&\le\sum_{i=1}^k\frac{tr\boldsymbol{A}_i^2}{tr\boldsymbol{A}_i}+\frac{tr\boldsymbol{A}_{k+1}^2}{tr\boldsymbol{A}_{k+1}}.
\end{aligned}\tag{4.10}$$

That is, the inequality holds when $n=k+1$. Thus we have proved. $\square$

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