Derivation of Equation (8)

The derivative of $L(\alpha_i)$ is given by

$$\frac{dL(\alpha_i)}{d\alpha_i} = \frac{(2a+2+s_ib-bq_i^2)\alpha_i^2 + (4as_i+5s_i+bs_i^2-q_i^2)\alpha_i + (2a+3)s_i^2}{2\alpha_i(1+b\alpha_i)(\alpha_i+s_i)^2}$$

Defining $\delta = 2a+2+s_ib-bq_i^2$, $\gamma = (4a+5)s_i+bs_i^2-q_i^2$ and $\eta = (2a+3)s_i^2$, we can write the numerator of the derivative as

$$J = \delta\alpha_i^2 + \gamma\alpha_i + \eta.$$

Define $\Delta = \gamma^2 - 4\delta\eta = \gamma^2 - 4\delta(2a+3)s_i^2$. If $\Delta > 0$, denote the two roots of $J = 0$ as $r_1 = \frac{-\gamma-\sqrt{\Delta}}{2\delta}$ and $\tilde{r}_1 = \frac{-\gamma+\sqrt{\Delta}}{2\delta}$. We assume that $b > 0$ so that the denominator of $dL(\alpha_i)/d\alpha_i$ is always greater than zero. We also assume that $a > -1.5$ so that $\eta > 0$. Consider the following three cases.

Case 1: $\delta > 0$

When $\Delta \leq 0$, $\frac{dL(\alpha_i)}{d\alpha_i} \geq 0$, which implies that $L(\alpha_i)$ is a nondecreasing function of α_i and thus is maximized at $\alpha_i^* = \infty$.

When $\Delta > 0$, $J = 0$ has two roots r_1 and $\tilde{r}_1$ defined earlier. If $\gamma > 0$, both r_1 and $\tilde{r}_1$ are less than zero, and thus $dL(\alpha_i)/d\alpha_i > 0$ for α_i, which implies that $L(\alpha_i)$ is maximized at $\alpha_i^* = \infty$. If $\gamma = 0$, then $r_1 < 0$ and $\tilde{r}_1 > 0$ and thus $dL(\alpha_i)/d\alpha_i > 0$ for $\alpha_i > \tilde{r}_1$, which implies $\alpha_i^* = \infty$. If $\gamma < 0$, both r_1 and $\tilde{r}_1$ are greater than zero. In this case, r_1 is a local maximum of $L(\alpha_i)$, r_1 is a local minimum of $L(\alpha_i)$, and $L(\alpha_i)$ is an increasing function of α_i for $\alpha_i > \tilde{r}_1$. Since $L(\infty) = 0$, the global maximum of $L(\alpha_i)$ in this case is achieved at $\alpha^* = r_1$ if $L(r_1) > 0$ or at $\alpha^* = \infty$ if $L(r_1) < 0$.

Case 2: $\delta = 0$

In this case, we have $J = \gamma\alpha_i + \eta$ and $r_2 = -(2a+3)s_i^2/\gamma$ is the root of $J = 0$. If $\gamma < 0$, $\alpha^* = r_2$ is the global maximum of $L(\alpha_i)$. If $\gamma \geq 0$, $L(\alpha_i)$ is an increasing function and $\alpha^* = \infty$ is the global maximum of $L(\alpha_i)$.

Case 3: $\delta < 0$

In this case, Δ is always greater than zero and thus $J = 0$ has two roots $r_1 > 0$ and $\tilde{r}_1 < 0$ and $dL(\alpha_i)/d\alpha_i < 0$ for $\alpha_i > r_1$. Therefore, $\alpha_i^* = r_1$ is the global maximum of $L(\alpha_i)$.

Summarizing the results in three cases, we get α_i^* given in (8).