

Derivation of joint distributions of (R, G) for p_n , p_u , and p_o

$$\begin{aligned}
 p_n(R, G) &= \\
 &= \frac{1}{RG} \int \frac{e^{-\theta \log(cR)} (\log(cR))^{a-1} \theta^a}{\Gamma(a)} \frac{e^{-\theta \log(G)} \log(G)^{a-1} \theta^a}{\Gamma(a)} \frac{e^{-\theta \gamma} \theta^{a_0-1} \gamma^{a_0}}{\Gamma(a_0)} d\theta \\
 &= \frac{1}{RG} \int \frac{e^{-\theta(\log(cR)+\log(G)+\log(\gamma))} (\log(cR)\log(G))^{a-1} \theta^{2a+a_0-1} \gamma^{a_0}}{\Gamma^2(a)\Gamma(a_0)} d\theta \\
 &= \frac{1}{RG} \int \frac{e^{-\theta(\log(cR)+\log(G)+\log(\gamma))} \theta^{2a+a_0-1} (\log(cR)+\log(G)+\gamma)^{2a+a_0}}{\Gamma(2a+a_0)} d\theta \cdot \left(\frac{\Gamma(2a+a_0) (\log(cR)\log(G))^{a-1} \gamma^{a_0}}{\Gamma^2(a)\Gamma(a_0)(\log(cR)+\log(G)+\gamma)^{2a+a_0}} \right) \\
 &= \frac{1}{RG} \frac{\Gamma(2a+a_0)}{\Gamma^2(a)\Gamma(a_0)} \frac{\gamma^{a_0} (\log(cR)\log(G))^{a-1}}{(\log(cR)+\log(G)+\gamma)^{2a+a_0}}
 \end{aligned}$$

$$\begin{aligned}
 p_u(R, G) &= \\
 &= \frac{1}{RG} \int \frac{e^{-\theta \log(cR)} (\log(cR))^{a-1} \theta^a}{\Gamma(a)} \frac{e^{-\theta \gamma_1} \theta^{a_0-1} \gamma_1^{a_0}}{\Gamma(a_0)} d\theta \cdot \int \frac{e^{-\theta \log(G)} (\log(G))^{a-1} \theta^a}{\Gamma(a)} \frac{e^{-\theta \gamma_2} \theta^{a_0-1} \gamma_2^{a_0}}{\Gamma(a_0)} d\theta \\
 &= \frac{1}{RG} \int \frac{e^{-\theta(\log(cR)+\gamma_1)} \theta^{a+a_0-1} (\log(cR)+\gamma_1)^{a+a_0}}{\Gamma(a+a_0)} d\theta \cdot \left(\frac{\Gamma(a+a_0) (\log(cR))^{a-1} \gamma_1^{a_0}}{\Gamma(a)\Gamma(a_0)(\log(cR)+\gamma_1)^{a+a_0}} \right) \\
 &\quad \cdot \int \frac{e^{-\theta(\log(G)+\gamma_2)} \theta^{a+a_0-1} (\log(G)+\gamma_2)^{a+a_0}}{\Gamma(a+a_0)} d\theta \cdot \left(\frac{\Gamma(a+a_0) (\log(G))^{a-1} \gamma_2^{a_0}}{\Gamma(a)\Gamma(a_0)(\log(G)+\gamma_2)^{a+a_0}} \right) \\
 &= \frac{1}{RG} \left(\frac{\Gamma(a+a_0)}{\Gamma(a)\Gamma(a_0)} \right)^2 \frac{(\gamma_1 \gamma_2)^{a_0} (\log(cR)\log(G))^{a-1}}{[(\log(cR)+\gamma_1)(\log(G)+\gamma_2)]^{a+a_0}}
 \end{aligned}$$

$$\begin{aligned}
 p_o(R, G) &= \\
 &= \frac{1}{RG} \int \frac{e^{-\theta \log(cR)} (\log(cR))^{a-1} \theta^a}{\Gamma(a)} \frac{e^{-\theta \gamma_2} \theta^{a_0-1} \gamma_2^{a_0}}{\Gamma(a_0)} d\theta \cdot \int \frac{e^{-\theta \log(G)} (\log(G))^{a-1} \theta^a}{\Gamma(a)} \frac{e^{-\theta \gamma_1} \theta^{a_0-1} \gamma_1^{a_0}}{\Gamma(a_0)} d\theta \\
 &= \frac{1}{RG} \int \frac{e^{-\theta(\log(cR)+\gamma_2)} \theta^{a+a_0-1} (\log(cR)+\gamma_2)^{a+a_0}}{\Gamma(a+a_0)} d\theta \cdot \left(\frac{\Gamma(a+a_0) (\log(cR))^{a-1} \gamma_2^{a_0}}{\Gamma(a)\Gamma(a_0)(\log(cR)+\gamma_2)^{a+a_0}} \right) \\
 &\quad \cdot \int \frac{e^{-\theta(\log(G)+\gamma_1)} \theta^{a+a_0-1} (\log(G)+\gamma_1)^{a+a_0}}{\Gamma(a+a_0)} d\theta \cdot \left(\frac{\Gamma(a+a_0) (\log(G))^{a-1} \gamma_1^{a_0}}{\Gamma(a)\Gamma(a_0)(\log(G)+\gamma_1)^{a+a_0}} \right) \\
 &= \frac{1}{RG} \left(\frac{\Gamma(a+a_0)}{\Gamma(a)\Gamma(a_0)} \right)^2 \frac{(\gamma_1 \gamma_2)^{a_0} (\log(cR)\log(G))^{a-1}}{[(\log(cR)+\gamma_2)(\log(G)+\gamma_1)]^{a+a_0}}
 \end{aligned}$$