

Additional file 1: Derivation of a preconditioned deflated coefficient matrix

Here we derive the preconditioned deflated coefficient matrix when the computational domain is divided such that some effects are included alone in a separate subdomain.

Consider the system of equations $\mathbf{C}\mathbf{x} = \mathbf{b}$ with a symmetric positive semi-definite coefficient

matrix $\mathbf{C} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix}$, a vector of solutions $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$, and a right-hand-side $\mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$.

Consider also the deflation-subspace matrix $\mathbf{Z}_d = \begin{bmatrix} \mathbf{Z}_{d_{11}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_z \end{bmatrix}$ where the submatrix $\mathbf{Z}_{d_{11}}$

contains only one non-zero element (which is equal to 1) per row, and the identity matrix $\mathbf{I}_z$

has the same size as $\mathbf{C}_{22}$. This deflation-subspace matrix $\mathbf{Z}_d$ corresponds to a division of a

computational domain $\mathbb{R}^n$ (with $\mathbf{x} \in \mathbb{R}^n$) where each entry of $\mathbf{x}_2$ is included alone in a subdomain.

The deflation matrix $\mathbf{P}$ is defined as $\mathbf{I} - \mathbf{C}\mathbf{Z}_d(\mathbf{Z}_d'\mathbf{C}\mathbf{Z}_d)^{-1}\mathbf{Z}_d'$.

Using a block-wise inversion, it can be shown that:

$$(\mathbf{Z}_d'\mathbf{C}\mathbf{Z}_d)^{-1} = \left(\begin{bmatrix} \mathbf{Z}_{d_{11}}' & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_z \end{bmatrix} \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{Z}_{d_{11}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_z \end{bmatrix} \right)^{-1} =$$

$$\begin{bmatrix} \mathbf{Q} & -\mathbf{Q}\mathbf{Z}_{d_{11}}'\mathbf{C}_{12}\mathbf{C}_{22}^{-1} \\ -\mathbf{C}_{22}^{-1}\mathbf{C}_{21}\mathbf{Z}_{d_{11}}\mathbf{Q} & \mathbf{T} \end{bmatrix}$$

with $\mathbf{Q} = (\mathbf{Z}_{d_{11}}'\mathbf{C}_{11}\mathbf{Z}_{d_{11}} - \mathbf{Z}_{d_{11}}'\mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21}\mathbf{Z}_{d_{11}})^{-1}$ and $\mathbf{T} = \mathbf{C}_{22}^{-1} +$

$$\mathbf{C}_{22}^{-1}\mathbf{C}_{21}\mathbf{Z}_{d_{11}}\mathbf{Q}\mathbf{Z}_{d_{11}}'\mathbf{C}_{12}\mathbf{C}_{22}^{-1}$$

After some algebra, it follows that the preconditioned deflated coefficient matrix $\mathbf{M}^{-1}\mathbf{P}\mathbf{C}$ is equal to:

$$\begin{aligned}
\mathbf{M}^{-1}\mathbf{P}\mathbf{C} &= \mathbf{M}^{-1}(\mathbf{I} - \mathbf{C}\mathbf{Z}_d(\mathbf{Z}_d'\mathbf{C}\mathbf{Z}_d)^{-1}\mathbf{Z}_d')\mathbf{C} \\
&= \mathbf{M}^{-1} \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \\
&\quad - \mathbf{M}^{-1} \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{Z}_{d_{11}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{Q} & -\mathbf{Q}\mathbf{Z}_{d_{11}}'\mathbf{C}_{12}\mathbf{C}_{22}^{-1} \\ -\mathbf{C}_{22}^{-1}\mathbf{C}_{21}\mathbf{Z}_{d_{11}}\mathbf{Q} & \mathbf{T} \end{bmatrix} \begin{bmatrix} \mathbf{Z}_{d_{11}}' & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \\
&= \mathbf{M}^{-1} \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \\
&\quad - \mathbf{M}^{-1} \begin{bmatrix} (\mathbf{C}_{11}\mathbf{Z}_{d_{11}}\mathbf{Q} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21}\mathbf{Z}_{d_{11}}\mathbf{Q})\mathbf{Z}_{d_{11}}'\mathbf{C}_{11} + (\mathbf{C}_{12}\mathbf{T} - \mathbf{C}_{11}\mathbf{Z}_{d_{11}}\mathbf{Q}\mathbf{Z}_{d_{11}}'\mathbf{C}_{12}\mathbf{C}_{22}^{-1})\mathbf{C}_{21} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \\
&= \mathbf{M}^{-1} \begin{bmatrix} \mathbf{C}_{11} - (\mathbf{C}_{11}\mathbf{Z}_{d_{11}}\mathbf{Q} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21}\mathbf{Z}_{d_{11}}\mathbf{Q})\mathbf{Z}_{d_{11}}'\mathbf{C}_{11} + (\mathbf{C}_{12}\mathbf{T} - \mathbf{C}_{11}\mathbf{Z}_{d_{11}}\mathbf{Q}\mathbf{Z}_{d_{11}}'\mathbf{C}_{12}\mathbf{C}_{22}^{-1})\mathbf{C}_{21} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \\
&= \mathbf{M}^{-1} \begin{bmatrix} \mathbf{S} - \mathbf{S}\mathbf{Z}_{d_{11}}(\mathbf{Z}_{d_{11}}'\mathbf{S}\mathbf{Z}_{d_{11}})^{-1}\mathbf{Z}_{d_{11}}'\mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}
\end{aligned}$$

with $\mathbf{S} = \mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{21}$.

This development proofs that the division of the ssSNPBLUP domain with 1 SNP effect per subdomain leads to a system matrix $\mathbf{M}^{-1}\mathbf{P}\mathbf{C}$ with zero entries for all equations associated with the SNP effects.