

Appendix

For an observational study that matches participants in pairs ($J = 2$), the value of L_s can be approximated by a GH quadrature

$$L_s \approx \sum_{\bar{\mathbf{q}}_k = (q_{1k}, q_{2k}, q_{3k})} \{l_{s_1}^{(1)}(q_{1k}, q_{3k})l_{s_1}^{(2)}(q_{1k}, q_{3k})\} \times \{l_{s_2}^{(1)}(q_{2k}, q_{3k})l_{s_2}^{(2)}(q_{2k}, q_{3k})\} \times w(\bar{\mathbf{q}}_k) \quad (\text{A.1})$$

where $\bar{\mathbf{q}}_k$ is one of the GH nodes and $w(\bar{\mathbf{q}}_k)$ is the weight for each node, generated by R function `mvQuad::createNIGrid`. We rewrite this GH quadrature approximation in matrices below to improve computation efficiency. For each matched set, the above equation can be rewritten as

$$L_s \approx \sum_{\bar{\mathbf{q}}_k = (q_{1k}, q_{2k}, q_{3k})} \{l_{s_1}^{(1)}(q_{1k}, q_{3k})l_{s_2}^{(1)}(q_{2k}, q_{3k})\} \times \{l_{s_1}^{(2)}(q_{1k}, q_{3k})l_{s_2}^{(2)}(q_{2k}, q_{3k})\} \times w(\bar{\mathbf{q}}_k) \quad (\text{A.2})$$

Let $l_{s_j}^{(1)}(a_{s_j}, b_s) = \prod_{t=1}^{T_{s_j}} l_{s_j t}(a_{s_j}, b_s)$, and

$$l_{s_j t}(a_{s_j}, b_s) = (1 - \pi_{s_j t})I(y_{s_j t} = 0) + \pi_{s_j t}f(y_{s_j t}|y_{s_j t} > 0)I(y_{s_j t} > 0). \quad (\text{A.3})$$

To calculate L_s for $s = 1, \dots, S$, we first denote

$$\bar{\mathbf{L}}^{(1)} = \left(\prod_{j=1}^2 l_{1_j}^{(1)}(q_{jk}, q_{3k}), \dots, \prod_{j=1}^2 l_{S_j}^{(1)}(q_{jk}, q_{3k}) \right)' \quad (\text{A.4})$$

then $\bar{\mathbf{L}}^{(1)}$ can be rewritten as

$$\ln \bar{\mathbf{L}}^{(1)} = \sum_{j=1}^2 A_j \bar{\mathbf{l}}_j(\bar{\mathbf{q}}_k) \quad (\text{A.5})$$

where

$$A_j = \begin{pmatrix} a_{1_j} & 0 & \cdots & 0 \\ 0 & a_{2_j} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a_{S_j} \end{pmatrix} \quad (\text{A.6})$$

$$a_{sj} = (1, \cdots, 1)_{1 \times T_{s_j}} \quad (\text{A.7})$$

$$\bar{\mathbf{l}}_j^{(1)}(\bar{\mathbf{q}}_k) = \begin{pmatrix} \ln(l_{1_j 1}(q_{jk}, q_{3k})) \\ \vdots \\ \ln(l_{1_j T_{1_j}}(q_{jk}, q_{3k})) \\ \vdots \\ \ln(l_{S_j 1}(q_{jk}, q_{3k})) \\ \vdots \\ \ln(l_{S_j T_{S_j}}(q_{jk}, q_{3k})) \end{pmatrix} \quad (\text{A.8})$$

We also denote

$$\bar{\mathbf{l}}^{(2)}(\bar{\mathbf{q}}_k) = \begin{pmatrix} \sum_{j=1}^2 \ln(l_{1_j}^{(2)}(q_{jk}, q_{3k})) \\ \vdots \\ \sum_{j=1}^2 \ln(l_{S_j}^{(2)}(q_{jk}, q_{3k})) \end{pmatrix} \quad (\text{A.9})$$

Let $\bar{\mathbf{Q}} = (\bar{\mathbf{q}}_1, \dots, \bar{\mathbf{q}}_K)$, then the matched-set-specific marginal likelihood can be approximated as

$$(L_1, \dots, L_S)' = \text{rowSum}\{\exp(\sum_{j=1}^2 A_j \bar{\mathbf{l}}_j^{(1)}(\bar{\mathbf{Q}}) + \bar{\mathbf{l}}^{(2)}(\bar{\mathbf{Q}})) \odot w(\bar{\mathbf{Q}})\} \quad (\text{A.10})$$

where $\odot$ represents element-wise multiplication of matrices and rowSum is the row-wise sum of elements in a matrix.