

Appendix A The Ricatti Equation

A.1 On the Solvability of the Ricatti Equation

In this section we will provide a procedure for transforming a *Ricatti*-type ODE in an analytically solvable linear equation if a special solution of the Ricatti ODE is known (for details on this subject see [1]).

For simplicity we let $p' = \frac{\partial}{\partial t}p(t, a)$ and rewrite (5) in the equivalent form

$$p' = (1 - p)(i - p\Delta m) \quad \Leftrightarrow \quad p' + (\Delta m + i)p + (-\Delta m)p^2 = i. \quad (\text{A1})$$

Now, for abbreviation let $g := \Delta m + i$ and $h := -\Delta m$ in (A1) to see that (5) indeed is an ODE of *Ricatti*-type:

$$p' + g(t, a)p + h(t, a)p^2 = i(t, a). \quad (\text{A2})$$

In order to transform (A2) into a linear ODE, we will carry out two transformations. First we note that $p_0(t, a) \equiv 1$ is obviously a special solution of (5). Then, we substitute

$$u := p - p_0 = p - 1 \quad (\text{A3})$$

in (A2) to derive at

$$u' + (g(t, a) + 2h(t, a))u + h(t, a)u^2 = 0 \quad (\text{A4})$$

which is a *Bernoulli*-type ODE, i.e.

$$u' + k(t)u + l(t)u^\alpha$$

with $\alpha = 2$. Finally, substituting

$$z := \frac{1}{u} \quad (\text{A5})$$

into (A4) leads to the linear ODE

$$z' - (g(t, a) + 2h(t, a))z = h(t, a), \quad (\text{A6})$$

which can be solved analytically by well known methods. Now, let $z = z(t, a)$ be a solution of (A6). Then the solution of (A2) can easily be calculated by reverse transformation:

$$p(t, a) = u(t, a) + 1 = 1 + \frac{1}{z(t, a)}. \quad (\text{A7})$$

References

- [1] Reid, W.T.: Scalar Riccati Differential Equations. In: Riccati Differential Equations, pp. 1–8. Elsevier Science, New York (1972)