Additional File 1

for

"Alternative tests and measures for between-study inconsistency in meta-analysis"

Derivation of the expectations of Q_γ statistics under the null hypothesis.

Under the null hypothesis of homogeneity, $Y_i \sim N(\mu, s_i^2)$ independently for $i = 1, \dots, k$. Recall that $Q_\gamma = \sum_{i=1}^k (|Y_i - \hat{\mu}_{\mathrm{CE}}|/s_i)^\gamma$, where $\hat{\mu}_{\mathrm{CE}} = \frac{\sum_{j=1}^k Y_j/s_j^2}{\sum_{j=1}^k 1/s_j^2}$. Let $X_i = (Y_i - \hat{\mu}_{\mathrm{CE}})/s_i$. We aim at deriving the γth moment of $|X_i|$ under the null, denoted by $m_{\gamma,i}$, so the expectation of Q_γ under the null is $E[Q_\gamma \mid H_0] = \sum_{i=1}^k m_{\gamma,i}$. Note that X_i follows a normal distribution with mean 0, so $|X_i|$ follows a folded normal distribution. Let σ_i^2 denote the variance of X_i. We have

$$
\begin{aligned}
\sigma_i^2 &= \mathrm{Var}(X_i) \\
&= \frac{1}{s_i^2}\,\mathrm{Var}(Y_i - M) \\
&= \frac{1}{s_i^2}\left(s_i^2 + \mathrm{Var}(M) - 2\,\mathrm{Cov}(y_i, M)\right) \\
&= \frac{1}{s_i^2}\left(s_i^2 + \frac{1}{\sum_{j=1}^k 1/s_j^2} - 2\,\mathrm{Cov}\left(y_i, \frac{\sum_{j=1}^k y_j/s_j^2}{\sum_{j=1}^k 1/s_j^2}\right)\right) \\
&= \frac{1}{s_i^2}\left(s_i^2 + \frac{1}{\sum_{j=1}^k 1/s_j^2} - 2\,\mathrm{Cov}\left(y_i, \frac{y_i/s_i^2}{\sum_{j=1}^N 1/s_j^2}\right)\right) \\
&= \frac{1}{s_i^2}\left(s_i^2 + \frac{1}{\sum_{j=1}^k 1/s_j^2} - \frac{2}{\sum_{j=1}^k 1/s_j^2}\right) \\
&= 1 - \frac{1}{s_i^2 \sum_{j=1}^k 1/s_j^2}.
\end{aligned}
$$

Based on Elandt (1961), the γ^{th} moment of folded normal distribution is:

$$
\begin{aligned}
m_{\gamma,i} = E[|X_i|^\gamma] &= \int_0^\infty \frac{2}{\sigma_i\sqrt{2\pi}} x_i^\gamma e^{-x_i^2/2\sigma_i^2}\, dx_i \\
&= \int_0^\infty \frac{2}{\sqrt{2\pi}} (y_i\sigma_i)^\gamma e^{-(y_i\sigma_i)^2/2\sigma_i^2}\, dy_i \\
&= 2\sigma_i^\gamma \int_0^\infty \frac{1}{\sqrt{2\pi}} y_i^\gamma e^{-y_i^2/2}\, dy_i \\
&= 2\sigma_i^\gamma \int_0^\infty \frac{1}{\sqrt{2\pi}} (2u)^{\gamma/2} e^{-u} \cdot (2u)^{-1/2}\, du \\
&= \frac{2^{(\gamma+1)/2}}{\sqrt{2\pi}} \sigma_i^\gamma \int_0^\infty u^{(\gamma+1)/2-1} e^{-u}\, du \\
&= \frac{2^{\gamma/2}}{\sqrt{2\pi}} \Gamma\left(\frac{\gamma+1}{2}\right) \sigma_i^\gamma
\end{aligned}
$$

where $\Gamma(\cdot)$ is the Gamma function.

Therefore, we have $E[Q_\gamma \mid H_0] = \sum_{i=1}^{k} m_{\gamma,i} = \frac{2^{\gamma/2}}{\sqrt{2\pi}}\Gamma\left(\frac{\gamma+1}{2}\right)\sum_{i=1}^{k}\sigma_i^{\gamma}$.

References

Elandt, R. C. (1961). The folded normal distribution: two methods of estimating parameters from moments. *Technometrics* **3,** 551–562.