

Supplementary Material

A.1. Proof of the positively invariant set

According to system (2)

$$\frac{dS_1}{dt}|_{S_1=0, S_2=0, V=0} = 0, \frac{dS_2}{dt}|_{S_1=0, S_2=0, V=0} = 0, \frac{dE}{dt}|_{S_1=0, S_2=0, E=0} = 0,$$

$$\frac{dI}{dt}|_{E=0, I=0} = 0, \frac{dV}{dt}|_{S_2=0, V=0} = 0, \frac{dB}{dt}|_{E=0, I=0, B=0} = 0, \frac{dS_h}{dt}|_{S_h=0} = 0, \frac{dI_{ah}}{dt}|_{S_h=0, I_{ah}=0} = 0,$$

all of the above values are non-negative in the bounded plane of $R_8^+ = \{(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7) : x_i \geq 0, i = 0, 1, 2, 3, 4, 5, 6, 7\}$. If the initial solution of system (2) is in R_8^+ , then all the solutions are non-negative. Let $(S_1, S_2, E, I, V, B, S_h, I_{ah})$ be the solution under the initial condition, satisfy $N = S_1 + S_2 + E + I + V, N_h = S_h + I_{ah}$, then

$$\frac{dN}{dt} = A + \alpha(S_2 + V) - \mu N - \gamma I \leq A - (\mu - \alpha)N.$$

It follows that

$$\limsup_{t \rightarrow \infty} N(t) = \frac{A}{\mu - \alpha}.$$

From system (2), one can obtain that

$$\frac{dB}{dt} \leq kN - (d + n\tau)B \leq \frac{kA}{\mu - \alpha} - (d + n\tau)B,$$

then

$$\limsup_{t \rightarrow \infty} B(t) = \frac{kA}{(\mu - \alpha)(d + n\tau)}.$$

Similarly, it can be obtained

$$\limsup_{t \rightarrow \infty} N_h = \frac{A_h}{\mu_h}.$$

This completes the proof.

A.2. Proof of the local stability of E^0

The Jacobian matrix of system (2) at E^0 is given by

$$J(E^0) = \begin{pmatrix} \Omega_1 & 0 & 0 \\ \Omega_2 & \Omega_3 & 0 \\ \Omega_4 & 0 & \Omega_5 \end{pmatrix}$$

where

$$F = \begin{pmatrix} \varepsilon\beta S_1^0 + \beta S_2^0 & \varepsilon\beta S_1^0 + \beta S_2^0 & \varepsilon\beta_1 S_1^0 + \beta_1 S_2^0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, V = \begin{pmatrix} \mu + \sigma & 0 & 0 \\ -\sigma & \mu + \gamma & 0 \\ -k & -k & d + n\tau \end{pmatrix},$$

Let $\Omega_1 = F - V$

$$\Omega_1 = \begin{pmatrix} \varepsilon\beta S_1^0 + \beta S_2^0 - (\mu + \sigma) & \varepsilon\beta S_1^0 + \beta S_2^0 & \varepsilon\beta_1 S_1^0 + \beta_1 S_2^0 \\ \sigma & -(\mu + \gamma) & 0 \\ k & k & -(d + n\tau) \end{pmatrix},$$

$$\Omega_2 = \begin{pmatrix} -\varepsilon\beta S_1^0 & -\varepsilon\beta S_1^0 & -\varepsilon\beta S_1^0 \\ -\beta_2 S_2^0 & -\beta_2 S_2^0 & -\beta_1 S_2^0 \\ 0 & 0 & 0 \end{pmatrix}, \Omega_3 = \begin{pmatrix} -(\mu + m) & \alpha & \alpha \\ m & -(\mu + \theta) & \delta \\ 0 & \theta & -(\mu + \delta) \end{pmatrix},$$

$$\Omega_4 = \begin{pmatrix} -\beta_h S_h^0 & -\beta_h S_h^0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \Omega_5 = \begin{pmatrix} -\mu_h & 0 \\ 0 & -(\mu_h + \delta_h + r) \end{pmatrix}.$$

Define $s(\Omega_1)$ be the spectral bound of matrix Ω_1 . $s(\Omega_1) = \max\{Re\lambda : \lambda \text{ is an eigenvalue of } \Omega_1\}$, according to Theorem 2 of [?], it can be obtained

$$R_0 > 1 \Leftrightarrow s(\Omega_1) > 0, R_0 < 1 \Leftrightarrow s(\Omega_1) < 0.$$

The characteristic equation of $J(\Omega_3)$ is

$$p(\lambda) = \lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0,$$

where

$$b_1 = 3\mu + \theta + m + \delta,$$

$$b_2 = \mu(\mu + \theta + \delta) + (\mu + m)(2\mu + \theta + \delta) - m\alpha,$$

$$b_3 = \mu(\mu + m)(\mu + \theta + \delta) - m\alpha(\mu + \theta + \delta).$$

Obviously, $b_1 > 0, b_2 > 0, b_3 > 0$, and

$$\begin{aligned} b_1 b_2 - b_3 &= (3\mu + \theta + \delta + m)[(\mu + \theta + \delta)(2\mu + m) + \mu(\mu + m) - m\alpha] \\ &\quad - (\mu + \theta + \delta)(\mu^2 + m\mu - m\alpha) \\ &> (3\mu + \theta + \delta + m)(\mu + \theta + \delta)(2\mu + m) - (\mu + \theta + \delta)(\mu^2 + m\mu) \\ &\quad + (\mu + \theta + \delta)m\alpha \\ &> 0. \end{aligned}$$

According to the Routh-Hurwitz criterion, the matrix $J(\Omega_3)$ has eigenvalues with negative real parts. Therefore, when $R_0 < 1$, there are $s(\Omega_1) < 0$ and $s(J|E^0) < 0$. Thus, the disease-free equilibrium E^0 of system (2) is locally asymptotically stable. When $R_0 > 1$, there is a positive eigenvalue, the disease-free equilibrium E^0 is unstable.

A.3. Proof of the global stability of E^0

First, we define the Lyapunov function as follows

$$\begin{aligned} L_1(t) = & S_1(t) - S_1^0 - S_1^0 \ln \frac{S_1(t)}{S_1^0} + S_2(t) - S_2^0 - S_2^0 \ln \frac{S_2(t)}{S_2^0} + V(t) - V^0 - V^0 \ln \frac{V(t)}{V^0} \\ & + E + \frac{(\mu + \sigma)(d + n\tau) - (d + n\tau)(\varepsilon\beta S_1^0 + \beta S_2^0) - k(\varepsilon\beta_1 S_1^0 + \beta_1 S_2^0)}{\sigma(d + n\tau)} I \\ & + \frac{\varepsilon\beta_1 S_1^0 + \beta_1 S_2^0}{d + n\tau} B. \end{aligned}$$

From equation (4), when $R_0 < 1$, it can be obtained

$$\begin{aligned} (\mu + \sigma)(d + n\tau) &> (d + n\tau)(\varepsilon\beta S_1^0 + \beta S_2^0) + k(\varepsilon\beta_1 S_1^0 + \beta_1 S_2^0), \\ \frac{(\mu + \sigma)(d + n\tau) - (d + n\tau)(\varepsilon\beta S_1^0 + \beta S_2^0) - k(\varepsilon\beta_1 S_1^0 + \beta_1 S_2^0)}{\sigma(d + n\tau)} &> 0, \end{aligned}$$

Taking derivative along with the solution of model (2), we have

$$\begin{aligned} \frac{dL_1(t)}{dt} = & (1 - \frac{S_1^0}{S_1})[A + \alpha(S_2 + V) - \varepsilon(\beta S_1 I + \beta S_1 E + \beta_1 S_1 B) \\ & - \frac{A + \alpha(S_2^0 + V^0)}{S_1^0} S_1] + (1 - \frac{S_2^0}{S_2})[mS_1 + \delta V - (\beta S_2 I + \beta S_2 E \\ & + \beta_1 S_2 B) - \frac{mS_1^0 + \delta V^0}{S_2^0} S_2] + (1 - \frac{V^0}{V})[\theta S_2 - \frac{\theta S_2^0}{V^0} V] \\ & + \varepsilon(\beta S_1 I + \beta S_1 E + \beta_1 S_1 B) + (\beta S_2 I + \beta S_2 E + \beta_1 S_2 B) - (\mu + \sigma)E \\ & + \frac{(\mu + \sigma)(d + n\tau) - (d + n\tau)(\varepsilon\beta S_1^0 + \beta S_2^0) - k(\varepsilon\beta_1 S_1^0 + \beta_1 S_2^0)}{\sigma(d + n\tau)} \\ & [\sigma E - (\mu + \gamma)I] + \frac{\varepsilon\beta_1 S_1^0 + \beta_1 S_2^0}{d + n\tau} [k(E + I) - (d + n\tau)B], \end{aligned} \quad (A1)$$

where $A = \mu S_1^0 + \mu V^0 + \mu S_2^0 - \alpha(S_2^0 + V^0)$, $mS_1^0 = \mu V^0 + \mu S_2^0$, $\theta S_2^0 = (\mu + \delta)V^0$, substituting into (A1), then

$$\begin{aligned} \frac{dL_1(t)}{dt} = & \mu S_1^0 (2 - \frac{S_1^0}{S_1} - \frac{S_1}{S_1^0}) + (\mu V^0 - \alpha V^0) (4 - \frac{S_1^0}{S_1} - \frac{S_2^0 S_1}{S_1^0 S_2} - \frac{V}{V^0} - \frac{S_2 V^0}{S_2^0 V}) \\ & + (\mu S_2^0 - \alpha S_2^0) (3 - \frac{S_1^0}{S_1} - \frac{S_2}{S_2^0} - \frac{S_2^0 S_1}{S_1^0 S_2}) + \delta V^0 (2 - \frac{S_2^0 V}{S_2^0 V^0} - \frac{S_2 V^0}{S_2^0 V}) \\ & + \alpha S_2^0 (2 - \frac{S_2^0 S_1}{S_1^0 S_2} - \frac{S_1^0 S_2}{S_2^0 S_1}) + \alpha V^0 (3 - \frac{S_2 V^0}{S_2^0 V} - \frac{S_1^0 V}{S_1 V^0} - \frac{S_2^0 S_1}{S_1^0 S_2}) \\ & + \frac{(\mu + \sigma)(\mu + \gamma)}{\sigma} I (R_0 - 1). \end{aligned}$$

Therefore, when $R_0 < 1$, $\frac{dL_1(t)}{dt} < 0$ and $\frac{dL_1(t)}{dt} = 0$ if and only if $S_1 = S_1^0, S_2 = S_2^0, E = 0, I = 0, V = V^0, B = 0$.

In order to prove the global stability of the disease-free equilibrium at E^0 , consider

system (2) with the remaining two equations in disease-free state,

$$\begin{cases} \frac{dS_h}{dt} = A_h - \mu_h S_h, \\ \frac{dI_{ah}}{dt} = -(\mu_h + \delta_h + r)I_{ah}. \end{cases}$$

Constructing the Lyapunov function as follows

$$L_2(t) = (S_h - S_h^0 - S_h^0 \ln \frac{S_h}{S_h^0}) + I_{ah}.$$

Taking the derivative at the disease-free equilibrium point $E_h^0 = (S_h^0, I_{ah}^0)$ of system (2), we can get

$$\begin{aligned} \frac{dL_2(t)}{dt} &= \frac{S_h - S_h^0}{S_h} (A_h - \mu_h S_h) - (\mu_h + \delta_h + r)I_{ah} \\ &= -\frac{A_h}{S_h} (S_h - S_h^0)^2 - (\mu_h + \delta_h + r)I_{ah}. \end{aligned}$$

It can be obtained $\frac{dL_2(t)}{dt} \leq 0$, and $\frac{dL_2(t)}{dt} = 0$ if and only if $S_h = S_h^0, I_{ah} = I_{ah}^0 = 0$. From the *LaSalle's* invariant principle, the disease-free equilibrium point $E_h^0 = (S_h^0, I_{ah}^0)$ is globally asymptotically stable within Γ .

In summary, when $R_0 < 1$, the disease-free equilibrium point E^0 of system (2) is globally asymptotically stable.

A.4. Proof of the global stability of E^*

The Jacobian matrix $J(E^*)$ at the endemic equilibrium E^* is

$$J(E^*) = \begin{pmatrix} \Psi_1 & \Psi_2 & 0 \\ \Psi_3 & \Psi_4 & 0 \\ \Psi_5 & \Psi_6 & \Psi_7 \end{pmatrix}$$

where

$$\Psi_1 = \begin{pmatrix} -\mu - m - \varepsilon(\beta I + \beta E + \beta_1 B) & \alpha & -\varepsilon\beta S_1 \\ m & -\theta - \mu - \beta(I + E) - \beta_1 B & -\beta S_2 \\ \varepsilon(\beta I + \beta E + \beta_1 B) & \beta(I + E) + \beta_1 B & -(\mu + \sigma) \end{pmatrix},$$

$$\Psi_2 = \begin{pmatrix} -\varepsilon\beta S_1 & \alpha & -\varepsilon\beta_1 S_1 \\ -\beta S_2 & \delta & -\beta_1 S_2 \\ \varepsilon\beta S_1 + \beta S_2 & 0 & \varepsilon\beta_1 S_1 + \beta_1 S_2 \end{pmatrix}, \Psi_3 = \begin{pmatrix} 0 & 0 & \sigma \\ 0 & \theta & 0 \\ 0 & 0 & k \end{pmatrix},$$

$$\Psi_4 = \begin{pmatrix} -(\mu + \gamma) & 0 & 0 \\ 0 & -(\mu + \sigma) & 0 \\ k & 0 & -(d + n\tau) \end{pmatrix}, \Psi_5 = \begin{pmatrix} 0 & 0 & -\beta_h S_h \\ 0 & 0 & \beta_h S_h \end{pmatrix},$$

$$\Psi_6 = \begin{pmatrix} -\beta_h S_h & 0 & 0 \\ \beta_h S_h & 0 & 0 \end{pmatrix}, \Psi_7 = \begin{pmatrix} -\beta_h(E + I) - \mu_h & 0 \\ \beta_h(E + I) & -(\mu_h + \delta_h + r) \end{pmatrix},$$

The characteristic equation of $J(\Psi_1)$ is

$$\begin{aligned}
|xI - J(\Psi_1)| &= (x + \mu + m + \varepsilon\beta I + \varepsilon\beta E + \varepsilon\beta_2 B)[x^2 + x(\beta^2 S_2 I + \beta^2 S_2 E + \beta\beta_1 S_2 B \\
&\quad - \varepsilon^2 S_1 I \beta^2 - \varepsilon^2 S_1 E \beta^2 - \varepsilon^2 \beta\beta_1 S_1 B - m\alpha) + (-\varepsilon^2 S_1 I \beta^2 - \varepsilon^2 S_1 E \beta^2 \\
&\quad - \varepsilon^2 \beta\beta_1 S_1 B)(\theta + \mu + \beta I + \beta E + \beta_1 B) + (\beta^2 S_2 I + \beta^2 S_2 E + \beta\beta_1 S_2 B) \\
&\quad (\mu + m + \varepsilon\beta I + \varepsilon\beta E + \varepsilon\beta B) + (\mu + \sigma)m\alpha] + (x + \theta + \mu + \beta I + \beta E \\
&\quad + \beta_1 B)(x + \mu + \sigma) \\
&= x^3 + a_2 x^2 + a_1 x + a_0
\end{aligned}$$

where

$$\begin{aligned}
a_2 &= \mu + m + \varepsilon\beta I + \varepsilon\beta E + \varepsilon\beta_2 B, \\
a_1 &= \beta^2 S_2 I + \beta^2 S_2 E + \beta\beta_1 S_2 B - \varepsilon^2 S_1 I \beta^2 - \varepsilon^2 S_1 E \beta^2 - \varepsilon^2 \beta\beta_1 S_1 B - m\alpha \\
&\quad + \mu + \sigma + \theta + \mu + \beta I + \beta E + \beta_1 B \\
a_0 &= (\beta^2 S_2 I + \beta^2 S_2 E + \beta\beta_1 S_2 B)(\mu + m + \varepsilon\beta I + \varepsilon\beta E + \varepsilon\beta B) \\
&\quad - \varepsilon^2 S_1 (I\beta^2 + E\beta^2 + \beta\beta_1 B)(\theta + \mu + \beta I + \beta E + \beta_1 B) \\
&\quad + (\mu + \sigma)m\alpha + (\theta + \mu + \beta I + \beta E + \beta_1 B)(\mu + \sigma)
\end{aligned}$$

Obviously, $a_0 > 0$, $a_1 > 0$, $a_2 > 0$, and

$$\begin{aligned}
a_1 a_2 - a_0 &= (\theta + \mu + \beta I + \beta E + \beta_1 B)(\varepsilon^2 S_1 I \beta^2 + \varepsilon^2 S_1 E \beta^2 + \varepsilon^2 \beta\beta_1 S_1 B + \mu + \sigma) \\
&\quad + (\mu + m + \varepsilon\beta I + \varepsilon\beta E + \varepsilon\beta B)(2\beta^2 S_2 I + 2\beta^2 S_2 E + 2\beta\beta_1 S_2 B - \varepsilon^2 S_1 I \beta^2 \\
&\quad + \varepsilon^2 S_1 E \beta^2 + \varepsilon^2 \beta\beta_1 S_1 B - m\alpha + 2\mu + \sigma + \theta + \beta I + \beta E + \beta_1 B) \\
&> 0.
\end{aligned}$$

According to the Routh-Hurwitz criterion, the matrix $J(\Psi_1)$ has eigenvalues with negative real parts. Thus, the endemic equilibrium E^* of system (2) is locally asymptotically stable.

Next, we define the Lyapunov function as follows

$$\begin{aligned}
L_3(t) &= S_1(t) - S_1^* - S_1^* \ln \frac{S_1(t)}{S_1^*} + S_2(t) - S_2^* - S_2^* \ln \frac{S_2(t)}{S_2^*} \\
&\quad + V(t) - V^* - V^* \ln \frac{V(t)}{V^*} + E(t) - E^* - E^* \ln \frac{E(t)}{E^*} \\
&\quad + \left(\frac{\varepsilon\beta S_1^* I^*}{\sigma E^*} + \frac{kI^*(\varepsilon\beta_1 S_1^* B^* + \beta_1 S_2^* B^*)}{\sigma E^*(kE^* + kI^*)} \right) [I(t) - I^* - I^* \ln \frac{I(t)}{I^*}] \\
&\quad + \frac{\varepsilon\beta_1 S_1^* B^* + \beta_1 S_2^* B^*}{kE^* + kI^*} [B(t) - B^* - B^* \ln \frac{B(t)}{B^*}]
\end{aligned}$$

Taking derivative along with the solution of model (2), we have

$$\begin{aligned}
\frac{dL_3(t)}{dt} = & (1 - \frac{S_1^*}{S_1})[A + \alpha(S_2 + V) - \varepsilon(\beta S_1 I + \beta S_1 E + \beta_1 S_1 B)] \\
& - (1 - \frac{S_1^*}{S_1})[\frac{A + \alpha(S_2^* + V^*) - \varepsilon(\beta S_1^* I^* + \beta S_1^* E + \beta_1 S_1^* B^*)}{S_1^*} S_1] \\
& + (1 - \frac{S_2^*}{S_2})[m S_1 + \delta V - (\beta S_2 I + \beta S_2 E + \beta_1 S_2 B)] \\
& - (1 - \frac{S_2^*}{S_2})[\frac{m S_1^* + \delta V^* - (\beta S_2^* I^* + \beta S_2^* E + \beta_1 S_2^* B^*)}{S_2^*} S_2] \\
& + (1 - \frac{E^*}{E})[\varepsilon(\beta S_1 I + \beta S_1 E + \beta_1 S_1 B) + \beta S_2 I + \beta S_2 E + \beta_1 S_2 B] \\
& - (1 - \frac{E^*}{E})[\frac{\varepsilon(\beta S_1^* I^* + \beta S_1^* E + \beta_1 S_1^* B^*) + \beta S_2^* I^* + \beta S_2^* E + \beta_1 S_2^* B^*}{E^*} E] \\
& + (1 - \frac{V^*}{V})[\theta S_2 - \frac{\theta S_2^*}{V^*} V] + (\varepsilon \beta S_1^* I^* + \beta S_2^* I^*)(1 - \frac{I^*}{I})(\frac{E}{E^*} - \frac{I}{I^*}) \\
& + \frac{k I^*(\varepsilon \beta_1 S_1^* B^* + \beta_1 S_2^* B^*)}{k E^* + k I^*}(1 - \frac{I^*}{I})(\frac{E}{E^*} - \frac{I}{I^*}) \\
& + \frac{\varepsilon \beta_1 S_1^* B^* + \beta_1 S_2^* B^*}{k E^* + k I^*} k E^*(1 - \frac{B^*}{B})(\frac{E}{E^*} - \frac{B}{B^*}) \\
& + \frac{\varepsilon \beta_1 S_1^* B^* + \beta_1 S_2^* B^*}{k E^* + k I^*} k I^*(1 - \frac{B^*}{B})(\frac{I}{I^*} - \frac{B}{B^*})
\end{aligned}$$

Taking note that

$$\theta S_2^* = (\mu + \delta) V^*, m S_1^* = \mu(S_2^* + V^*) - \delta V^* + \beta S_2^* I^* + \beta S_2^* E^* + \beta_1 S_2^* B^*$$

$$A = \mu(S_1^* + S_2^* + V^*) - \alpha(S_2^* + V^*) + \varepsilon(\beta S_1^* I^* + \beta S_1^* E^* + \beta_1 S_1^* B^*) + \beta S_2^* I^* + \beta S_2^* E^* + \beta_1 S_2^* B^*.$$

Then

$$\begin{aligned}
\frac{dL_3(t)}{dt} &= \mu S_1^* (2 - \frac{S_1^*}{S_1} - \frac{S_1}{S_1^*}) + \mu V^* (4 - \frac{S_1^*}{S_1} - \frac{S_2^* S_1}{S_1^* S_2} - \frac{V}{V^*} - \frac{S_2 V^*}{S_2^* V}) \\
&\quad + \mu S_2^* (3 - \frac{S_1^*}{S_1} - \frac{S_2}{S_2^*} - \frac{S_2^* S_1}{S_1^* S_2}) + \delta V^* (2 - \frac{S_2^* V}{S_2 V^*} - \frac{S_2 V^*}{S_2^* V}) \\
&\quad - \alpha S_2^* (1 - \frac{S_1^*}{S_1} - \frac{S_2}{S_2^*} + \frac{S_2^* S_1}{S_1^* S_2}) - \alpha V^* (1 - \frac{S_1^*}{S_1} - \frac{V}{V^*} + \frac{S_1^* V}{S_1 V^*}) \\
&\quad + \varepsilon \beta S_1^* I^* (3 - \frac{S_1^*}{S_1} - \frac{I^* E}{I E^*} - \frac{E^* S_1 I}{E S_1^* I^*}) + \varepsilon \beta S_1^* E^* (2 - \frac{S_1^*}{S_1} - \frac{S_1}{S_1^*}) \\
&\quad + \varepsilon \beta_1 S_1^* B^* (2 - \frac{S_1^*}{S_1} - \frac{E}{E^*} + \frac{B}{B^*} - \frac{E^* S_1 B}{E S_1^* B^*}) + \beta S_2^* E^* (3 - \frac{S_1^*}{S_1} - \frac{S_1 S_2^*}{S_2 S_1^*} - \frac{S_2}{S_2^*}) \\
&\quad + \beta S_2^* I^* (4 - \frac{S_1^*}{S_1} - \frac{I^* E}{I E^*} - \frac{S_1 S_2^*}{S_2 S_1^*} - \frac{E^* S_2 I}{E S_2^* I^*}) \\
&\quad + \beta_1 S_2^* B^* (3 - \frac{S_1^*}{S_1} - \frac{S_1 S_2^*}{S_2 S_1^*} - \frac{E}{E^*} + \frac{B}{B^*} - \frac{E^* S_2 B}{E S_2^* B^*}) \\
&\quad + \frac{k I^* \varepsilon \beta_1 S_1^* B^*}{k E^* + k I^*} (2 + \frac{E}{E^*} - \frac{B}{B^*} - \frac{I^* E}{I E^*} - \frac{B^* I}{B I^*}) \\
&\quad + \frac{k I^* \beta_1 S_2^* B^*}{k E^* + k I^*} (2 + \frac{E}{E^*} - \frac{B}{B^*} - \frac{I^* E}{I E^*} - \frac{B^* I}{B I^*}) \\
&\quad + \frac{k E^* \varepsilon \beta_1 S_1^* B^*}{k E^* + k I^*} (\frac{E}{E^*} - \frac{B}{B^*} - \frac{B^* E}{B E^*} + 1) + \frac{k E^* \beta_1 S_2^* B^*}{k E^* + k I^*} (\frac{E}{E^*} - \frac{B}{B^*} - \frac{B^* E}{B E^*} + 1) \\
&= \mu S_1^* (2 - \frac{S_1^*}{S_1} - \frac{S_1}{S_1^*}) + (\mu V^* - \alpha V^*) (4 - \frac{S_1^*}{S_1} - \frac{S_2^* S_1}{S_1^* S_2} - \frac{V}{V^*} - \frac{S_2 V^*}{S_2^* V}) \\
&\quad + (\mu S_2^* - \alpha S_2^*) (3 - \frac{S_1^*}{S_1} - \frac{S_2}{S_2^*} - \frac{S_2^* S_1}{S_1^* S_2}) + \delta V^* (2 - \frac{S_2^* V}{S_2 V^*} - \frac{S_2 V^*}{S_2^* V}) \\
&\quad + \alpha S_2^* (2 - \frac{S_2^* S_1}{S_1^* S_2} - \frac{S_2^* S_1}{S_1^* S_2}) + \alpha V^* (3 - \frac{S_2^* S_1}{S_1^* S_2} - \frac{S_1^* V}{V^* S_1} - \frac{S_2 V^*}{S_2^* V}) \\
&\quad + \varepsilon \beta S_1^* I^* (3 - \frac{S_1^*}{S_1} - \frac{I^* E}{I E^*} - \frac{E^* S_1 I}{E S_1^* I^*}) + \varepsilon \beta S_1^* E^* (2 - \frac{S_1^*}{S_1} - \frac{S_1}{S_1^*}) \\
&\quad + \beta S_2^* I^* (4 - \frac{S_1^*}{S_1} - \frac{I^* E}{I E^*} - \frac{S_1 S_2^*}{S_2 S_1^*} - \frac{E^* S_2 I}{E S_2^* I^*}) + \beta S_2^* E^* (3 - \frac{S_1^*}{S_1} - \frac{S_1 S_2^*}{S_2 S_1^*} - \frac{S_2}{S_2^*}) \\
&\quad + \frac{k I^* \varepsilon \beta_1 S_1^* B^*}{k E^* + k I^*} (4 - \frac{S_1^*}{S_1} - \frac{I^* E}{I E^*} - \frac{B^* I}{B I^*} - \frac{E^* S_1 B}{E S_1^* B^*}) \\
&\quad + \frac{k I^* \beta_1 S_2^* B^*}{k E^* + k I^*} (5 - \frac{S_1^*}{S_1} - \frac{S_1 S_2^*}{S_2 S_1^*} - \frac{I^* E}{I E^*} - \frac{B^* I}{B I^*} - \frac{E^* S_2 B}{E S_2^* B^*}) \\
&\quad + \frac{k E^* \varepsilon \beta_1 S_1^* B^*}{k E^* + k I^*} (3 - \frac{S_1^*}{S_1} - \frac{B^* E}{B E^*} - \frac{E^* S_1 B}{E S_1^* B^*}) \\
&\quad + \frac{k E^* \beta_1 S_2^* B^*}{k E^* + k I^*} (4 - \frac{S_1^*}{S_1} - \frac{S_1 S_2^*}{S_2 S_1^*} - \frac{B^* E}{B E^*} - \frac{E^* S_2 B}{E S_2^* B^*}).
\end{aligned}$$

Since

$$\begin{aligned}
2 - \frac{S_1^*}{S_1} - \frac{S_1}{S_1^*} &\leq 0, 3 - \frac{S_1^*}{S_1} - \frac{I^* E}{I E^*} - \frac{E^* S_1 I}{E S_1^* I^*} \leq 0 \\
4 - \frac{S_1^*}{S_1} - \frac{I^* E}{I E^*} - \frac{S_1 S_2^*}{S_2 S_1^*} - \frac{E^* S_2 I}{E S_2^* I^*} &\leq 0, 5 - \frac{S_1^*}{S_1} - \frac{S_1 S_2^*}{S_2 S_1^*} - \frac{I^* E}{I E^*} - \frac{B^* I}{B I^*} - \frac{E^* S_2 B}{E S_2^* B^*} \leq 0.
\end{aligned}$$

Therefore $\frac{dL_3(t)}{dt} \leq 0$, $\frac{dL_3(t)}{dt} = 0$ if and only if $S_1 = S_1^*, S_2 = S_2^*, E = E^*, I = I^*, V = V^*, B = B^*$. In order to prove the global stability of the endemic equilibrium E^* , we need to consider the systems of the remaining two equations of system (2) in the endemic state

$$\begin{cases} \frac{dS_h}{dt} = A_h - \beta_h S_h (E^* + I^*) - \mu_h S_h, \\ \frac{dI_{ah}}{dt} = \beta_h S_h (E^* + I^*) - (\mu_h + \delta_h + r) I_{ah}. \end{cases}$$

We define the Lyapunov function as follows

$$L_4(t) = S_h - S_h^* - S_h^* \ln \frac{S_h}{S_h^*} + I_{ah} - I_{ah}^* - I_{ah}^* \ln \frac{I_{ah}}{I_{ah}^*},$$

Taking the derivative at the endemic equilibrium point E^* of system (2), we can get

$$\begin{aligned} \frac{dL_4(t)}{dt} &= \left(1 - \frac{S_h^*}{S_h}\right) [\beta_h S_h^* (E + I) + \mu_h S_h^* - \beta_h S_h (E^* + I^*) - \mu_h S_h] \\ &\quad + \left(1 - \frac{I_{ah}^*}{I_{ah}}\right) [\beta_h S_h (E^* + I^*) - (\mu_h + \delta_h + r) I_{ah}] \\ &= -\frac{\beta_h (E^* + I^*) + \mu_h}{S_h} (S_h - S_h^*)^2 - \frac{\mu_h + \delta_h + r}{I_{ah}} (I_{ah} - I_{ah}^*)^2. \end{aligned}$$

Therefore $\frac{dL_4(t)}{dt} \leq 0$, $\frac{dL_4(t)}{dt} = 0$ if and only if $S_h = S_h^*, I_{ah} = I_{ah}^*$. From the *LaSalle's* invariant principle, the endemic equilibrium point E^* is globally asymptotically stable within Γ .