Supplementary Material

Appendix: Derivation of F and V matrices

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Using Next-generation matrices;

Let the infective state vector be

$$\mathbf{x} = (E_{H1}, E_{H2}, I_H, T_H, E_M, I_M)^\top.$$

Following the next-generation method, write the infective subsystem as

$$\dot{\mathbf{x}} = \mathcal{F}(\mathbf{x}) - \mathcal{V}(\mathbf{x}),$$

where $\mathcal{F}$ collects new-infection terms and $\mathcal{V}$ contains transitions and removals. Evaluating Jacobians at the DFE E^0 yields the matrices $F = [\partial \mathcal{F}_i / \partial x_j]_{E^0}$ and $V = [\partial \mathcal{V}_i / \partial x_j]_{E^0}$.

With $S_H^0 = \pi/\mu_1$ and $S_M^0 = \Lambda/\mu_2$, the transmission (new-infection) Jacobian is

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \eta_S S_H^0 \\ 0 & 0 & \xi_S S_H^0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho S_M^0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The transition matrix V (transfers out minus transfers in, evaluated at the DFE) is

$$V = \begin{pmatrix} \mu_1 + \tau_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_1 + \tau_2 & 0 & 0 & 0 & 0 \\ -\tau_1 & -\tau_2 & \mu_1 + \delta_1 + \sigma & 0 & 0 & 0 \\ 0 & 0 & -\sigma & \mu_1 + \delta_1 + \omega & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_2 + \phi & 0 \\ 0 & 0 & 0 & 0 & -\phi & \mu_2 \end{pmatrix}.$$