

Appendix:

A.1 Local stability

The Jacobi matrix A of system (2) is as following,

$$\begin{pmatrix} r - 2a_{11}x - \frac{a_{12}y_1}{1+mx+ny_1} + \frac{ma_{12}xy_1}{(1+mx+ny_1)^2} & -\frac{a_{12}x(1+mx)}{(1+mx+ny_1)^2} & -d_2 - 2a_{22}y_1 + \frac{a_{21}x}{(1+mx+ny_1)} - \frac{a_{21}nxy_1}{(1+mx+ny_1)^2} - \beta y_2 \\ \frac{a_{21}y_1(1+ny_1)}{(1+mx+ny_1)^2} & 0 & \beta y_2 \\ 0 & 0 & 0 \\ -\beta y_1 & 0 & 0 \\ \beta y_1 - d_3 - 2a_{33}y_2 & 0 & 0 \end{pmatrix}.$$

By calculating the characteristic equation $\det(A - \lambda I) = 0$ of equilibria, we get the following conclusions.

Theorem A.1 *The trivial solution $O(0, 0, 0)$ of system (2) is an unstable saddle; Let $R_1 = \frac{a_{21}r}{d_2(a_{11}+mr)}$, then the boundary equilibrium $E_1(\frac{r}{a_{11}}, 0, 0)$ of system (2) is locally asymptotic stable when $R_1 < 1$, otherwise, E_1 is unstable, at the same time, equilibrium $E_2(\hat{x}, \hat{y}_1, 0)$ exists.*

For the equilibrium $E_2(\hat{x}, \hat{y}_1, 0)$, characteristic equation is obtained:

$$\begin{vmatrix} -a_{11}\hat{x} + \frac{ma_{12}\hat{x}\hat{y}_1}{(1+m\hat{x}+n\hat{y}_1)^2} - \lambda & \frac{-a_{12}\hat{x}(1+m\hat{x})}{(1+m\hat{x}+n\hat{y}_1)^2} \\ \frac{a_{21}\hat{y}_1(1+n\hat{y}_1)}{(1+m\hat{x}+n\hat{y}_1)^2} & -a_{22}\hat{y}_1 - \frac{a_{21}n\hat{x}\hat{y}_1}{(1+m\hat{x}+n\hat{y}_1)^2} - \lambda \end{vmatrix} (\beta\hat{y}_1 - d_3 - \lambda) = 0.$$

$\lambda_3 = \beta\hat{y}_1 - d_3$, when $R_0 < 1$, we have $\kappa < \frac{d_3}{\beta}$, thus, $\lambda_3 < 0$.

Assuming that

$$\begin{aligned} \omega_1 &= -a_{11}\hat{x} + \frac{ma_{12}\hat{x}\hat{y}_1}{(1+m\hat{x}+n\hat{y}_1)^2}, \quad \omega_2 = \frac{-a_{12}\hat{x}(1+m\hat{x})}{(1+m\hat{x}+n\hat{y}_1)^2}, \\ \omega_3 &= \frac{a_{21}\hat{y}_1(1+n\hat{y}_1)}{(1+m\hat{x}+n\hat{y}_1)^2}, \quad \omega_4 = -a_{22}\hat{y}_1 - \frac{a_{21}n\hat{x}\hat{y}_1}{(1+m\hat{x}+n\hat{y}_1)^2}, \end{aligned}$$

then the other two characteristic roots satisfy the following equation,

$$\lambda^2 - (\omega_1 + \omega_4)\lambda + \omega_1\omega_4 - \omega_2\omega_3 = 0. \quad (B1)$$

Here $\omega_2 < 0, \omega_3 > 0, \omega_4 < 0$, if $\omega_1 < 0$, then, $\omega_1 + \omega_4 < 0$, $\omega_1\omega_4 - \omega_2\omega_3 > 0$. At this time, equation (B1) will have two characteristic roots with negative real part.

By virtue of $\omega_1 < 0 \Leftrightarrow \frac{ma_{12}\hat{x}\hat{y}_1}{(1+m\hat{x}+n\hat{y}_1)^2} < a_{11}\hat{x}$, and $1 + m\hat{x} + n\hat{y}_1 = \frac{a_{12}\hat{y}_1}{r - a_{11}\hat{x}}$, $\hat{y}_1 = \frac{(1+m\hat{x})(r-a_{11}\hat{x})}{a_{12}-nr+na_{11}\hat{x}}$, therefore,

$$\omega_1 < 0 \Leftrightarrow \frac{a_{11}a_{12}(1+m\hat{x})(r-a_{11}\hat{x})}{a_{12}-nr+na_{11}\hat{x}} > m(r-a_{11}\hat{x})^2,$$

obviously, $r - a_{11}\hat{x} > 0$, in addition, according to the existence of $E_2(\hat{x}, \hat{y}_1, 0)$, we know $a_{12} - nr + na_{11}\hat{x} > 0$, thus, the above inequality can be turned into the form below,

$$mna_{11}^2\hat{x}^2 + 2a_{11}m(a_{12} - nr)\hat{x} + a_{11}a_{12} + mnr^2 - a_{12}mr > 0,$$

$\Delta_1 = 4a_{11}^2 a_{12} m(ma_{12} - mn r - na_{11})$, it is easy to see that $\Delta_1 < 0$ and $\omega_1 < 0$ when $ma_{12} < n(mr + a_{11})$. As a result, about the stability of $E_2(\hat{x}, \hat{y}_1, 0)$, we have the following results.

Theorem A.2 *Assuming $a_{21}r > d_2(a_{11} + mr)$, if $ma_{12} < n(mr + a_{11})$ and $R_0 < 1$, the disease-free equilibrium $E_2(\hat{x}, \hat{y}_1, 0)$ is locally asymptotic stable.*

In the following, the local stability of the positive equilibrium $E^*(x^*, y_1^*, y_2^*)$ is discussed, the corresponding characteristic equation of E^* is

$$\begin{vmatrix} -a_{11}x^* + \frac{ma_{12}x^*y_1^*}{(1+mx^*+ny_1^*)^2} - \lambda & \frac{-a_{12}x^*(1+mx^*)}{(1+mx^*+ny_1^*)^2} & 0 \\ \frac{a_{21}y_1^*(1+ny_1^*)}{(1+mx^*+ny_1^*)^2} & -a_{22}y_1^* - \frac{a_{21}nx^*y_1^*}{(1+mx^*+ny_1^*)^2} - \lambda & -\beta y_1^* \\ 0 & \beta y_2^* & -a_{33}y_2^* - \lambda \end{vmatrix} = 0. \quad (B2)$$

Let

$$p_1 = -a_{11}x^* + \frac{ma_{12}x^*y_1^*}{(1+mx^*+ny_1^*)^2}, \quad p_2 = \frac{-a_{12}x^*(1+mx^*)}{(1+mx^*+ny_1^*)^2},$$

$$p_3 = \frac{a_{21}y_1^*(1+ny_1^*)}{(1+mx^*+ny_1^*)^2}, \quad p_4 = -a_{22}y_1^* - \frac{a_{21}nx^*y_1^*}{(1+mx^*+ny_1^*)^2},$$

then (B2) can be simplified as,

$$(-a_{33}y_2^* - \lambda) \begin{vmatrix} p_1 - \lambda & p_2 \\ p_3 & p_4 - \lambda \end{vmatrix} - \beta y_2^* \begin{vmatrix} p_1 - \lambda & 0 \\ p_3 & -\beta y_1^* \end{vmatrix} = 0.$$

the above expression turns into:

$$\lambda^3 + (a_{33}y_2^* - (p_1 + p_4))\lambda^2 + [(p_1p_4 - p_2p_3) + \beta^2 y_1^* y_2^* - a_{33}y_2^*(p_1 + p_4)]\lambda + a_{33}y_2^*(p_1p_4 - p_2p_3) - \beta^2 y_1^* y_2^* p_1 = 0.$$

According to Hurwitz rules, if the characteristic roots of the equation have negative real parts, it should satisfy:

$$a_{33}y_2^* - (p_1 + p_4) > 0, \quad a_{33}y_2^*(p_1p_4 - p_2p_3) - \beta^2 y_1^* y_2^* p_1 > 0, \quad (B3)$$

$$[a_{33}y_2^* - (p_1 + p_4)][(p_1p_4 - p_2p_3) + \beta^2 y_1^* y_2^* - a_{33}y_2^*(p_1 + p_4)] - a_{33}y_2^*(p_1p_4 - p_2p_3) + \beta^2 y_1^* y_2^* p_1 > 0. \quad (B4)$$

the expression of (B4) is equivalent to

$$\beta^2 a_{33}y_1^*(y_2^*)^2 - a_{33}^2(y_2^*)^2(p_1 + p_4) - (p_1 + p_4)(p_1p_4 - p_2p_3) - p_4\beta^2 y_1^* y_2^* + a_{33}y_2^*(p_1 + p_4)^2 > 0. \quad (B5)$$

Here $p_2 < 0, p_3 > 0, p_4 < 0$, If $p_1 < 0$, then $p_1 + p_4 < 0, p_1p_4 - p_2p_3 > 0$, the inequalities (B3) and (B5) must be verified.

Because of $p_1 = -a_{11}x^* + \frac{ma_{12}x^*y_1^*}{(1+mx^*+ny_1^*)^2} < 0 \Leftrightarrow \frac{ma_{12}y_1^*}{(1+mx^*+ny_1^*)^2} < a_{11}$, here $1+mx^*+ny_1^* = \frac{a_{12}y_1^*}{r-a_{11}x^*}$, $r - a_{11}x^* > 0$, then the above inequality $p_1 < 0$ is inequivalent to

$$a_{11}a_{12}y_1^* > m(r - a_{11}x^*)^2, \quad (B6)$$

here $y_1^* = \frac{(r-a_{11}x^*)(1+mx^*)}{a_{12}-nr+na_{11}x^*}$, (B6) can be reduced to,

$$\frac{a_{11}a_{12}(1+mx^*)(r-a_{11}x^*)}{a_{12}-nr+na_{11}x^*} > m(r-a_{11}x^*)^2, \quad (B7)$$

On the basis of existence of E^* , we have $a_{12}-nr+na_{11}x^* > 0$, $r-a_{11}x^* > 0$. (B7) is true, it only needs to meet

$$mna_{11}^2(x^*)^2 + 2a_{11}m(a_{12}-nr)x^* + a_{11}a_{12} + mn r^2 - a_{12}mr > 0, \quad (B8)$$

$\Delta_2 = 4a_{11}^2[a_{12}m(ma_{12}-nrm-na_{11})]$, when $ma_{12} < n(mr+a_{11})$, $\Delta_2 < 0$, (B8) must be true, so the following conclusion can be gotten:

Theorem A.3 *If $R_0 > 1$, the positive equilibrium $E^*(x^*, y_1^*, y_2^*)$ of system (2) is locally asymptotic stable when $ma_{12} < n(mr+a_{11})$.*

A.2 Global stability

According to Theorem 5 and Theorem 6, conclusions about equilibria E_1 and E_2 can be derived.

Corollary 1 *If $a_{21}r < a_{11}d_2$, the boundary equilibrium $E_1 (\frac{r}{a_{11}}, 0, 0)$ of system (2) is globally asymptotic stable.*

Corollary 2 *Assume $a_{21}r > d_2(a_{11}+mr)$, the disease-free equilibrium $E_2 (\hat{x}, \hat{y}_1, 0)$ of system (2) is globally asymptotic stable when $a_{11} > m(r-a_{11}\hat{x})$ and $R_0 < 1$.*

Remark A.1 *Because of the same reason as Remark 6 of Theorem 6, condition $a_{11} > m(r-a_{11}\hat{x})$ in Corollary 2 can be replaced by $a_{12} < \min\{\frac{a_{11}n}{m}, nr\}$, which is easily checked.*

About the positive equilibrium E^* , according to Theorem 7, its global stability conditions is also gotten.

Corollary 3 *Assume $R_0 > 1$, when $a_{11} > m(r-a_{11}x^*)$, the positive equilibrium $E^*(x^*, y_1^*, y_2^*)$ of system (2) is globally asymptotic stable.*

Remark A.2 *Similarly, the condition $a_{11} > m(r-a_{11}x^*)$ in Corollary 3 can also use condition $a_{12} < \min\{\frac{na_{11}}{m}, nr\}$ instead which is easy to verify.*

A.3 Permanence

In the following, we give sufficient conditions of permanence for system (2).

Theorem A.4 *Assuming that $r > \frac{a_{12}}{n}$, $\beta a_{21}r > \beta a_{11}d_2 + a_{11}a_{22}d_3$ and*

$$\begin{aligned} a_{21}a_{33}\beta\frac{1}{a_{11}}(r-\frac{a_{12}}{n}) &> n\beta^3\frac{(a_{21}r-a_{11}d_2)^2}{a_{11}^2a_{22}^2} + [\frac{m\beta^3r}{a_{11}} + \beta^3 + An]\frac{(a_{21}r-a_{11}d_2)}{a_{11}a_{22}} \\ &+ Am\frac{r}{a_{11}} + A, \end{aligned}$$

are verified, here $A = a_{33}d_2\beta + d_3a_{22}a_{33} - \beta^2d_3$, then at this time system (2) is permanent.

Proof. By virtue of the comparison theorem, for system (2), it is easy to see

$$\limsup_{t \rightarrow +\infty} x(t) \leq \frac{r}{a_{11}} =: \bar{K}.$$

When $r > \frac{a_{12}}{n}$,

$$\liminf_{t \rightarrow +\infty} x(t) \geq \frac{1}{a_{11}}(r - \frac{a_{12}}{n}) =: \underline{K}.$$

In addition, we have

$$y_1'(t) \leq y_1(t)[-d_2 - a_{22}y_1 + a_{21}x(t)],$$

then if $a_{21}r > a_{11}d_2$,

$$\limsup_{t \rightarrow +\infty} y_1(t) \leq \frac{a_{21}r - a_{11}d_2}{a_{11}a_{22}} =: \bar{K}_1.$$

for sufficient large $t > 0$,

$$y_2'(t) \leq y_2(t)[\beta\bar{K}_1 - d_3 - a_{33}y_2],$$

In the same way, when $\beta a_{21}r > \beta a_{11}d_2 + a_{11}a_{22}d_3$,

$$\limsup_{t \rightarrow +\infty} y_2(t) \leq \frac{\beta\bar{K}_1 - d_3}{a_{33}} = \frac{\beta a_{21}r - \beta a_{11}d_2 - a_{11}a_{22}d_3}{a_{11}a_{22}a_{33}} =: \bar{K}_2.$$

Assuming t large enough,

$$y_1'(t) \geq y_1(t)[-d_2 - a_{22}y_1 + \frac{a_{21}\underline{K}}{1 + m\bar{K} + n\bar{K}_1} - \beta\bar{K}_2],$$

According to the comparison theorem, if $\frac{a_{21}\underline{K}}{1 + m\bar{K} + n\bar{K}_1} > \beta\bar{K}_2 + d_2$,

$$\liminf_{t \rightarrow +\infty} y_1(t) \geq (\frac{a_{21}\underline{K}}{1 + m\bar{K} + n\bar{K}_1} - \beta\bar{K}_2 - d_2) \frac{1}{a_{22}} =: \underline{K}_1.$$

Similarly, when $\underline{K}_1 > \frac{d_3}{\beta}$,

$$\liminf_{t \rightarrow +\infty} y_2(t) \geq \frac{\beta\underline{K}_1 - d_3}{a_{33}} =: \underline{K}_2.$$

Here $\underline{K}_1 > \frac{d_3}{\beta}$ is equivalent to

$$\frac{a_{21}\underline{K}}{1 + m\bar{K} + n\bar{K}_1} > \frac{\beta^3\bar{K}_1 - \beta^2d_3 + a_{33}d_2\beta + d_3a_{22}a_{33}}{a_{33}\beta}.$$

Substituting $\underline{K}$, $\bar{K}$ and $\bar{K}_1$ into the above inequality, we find that $\underline{K}_1 > \frac{d_3}{\beta}$ equals to

$$\begin{aligned} a_{21}a_{33}\beta \frac{1}{a_{11}}(r - \frac{a_{12}}{n}) &> n\beta^3 \frac{(a_{21}r - a_{11}d_2)^2}{a_{11}^2a_{22}^2} + [\frac{m\beta^3r}{a_{11}} + \beta^3 + An] \frac{(a_{21}r - a_{11}d_2)}{a_{11}a_{22}}, \\ &+ Am \frac{r}{a_{11}} + A, \end{aligned}$$

here $A = a_{33}d_2\beta + d_3a_{22}a_{33} - \beta^2d_3$, the theorem is proofed.