Estimation of satellite tag pop-up location error

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Argos location data include the semi-major axis length a, semi-minor axis length b, and angle of orientation θ of the $\sqrt{2}\sigma$ (63% probability) ellipse of the bivariate normal error distribution $N_2(0, \Sigma)$ associated with each estimated geographic location. The covariance matrix Σ of the error distribution can be recovered from $a, b,$ and θ based on its spectral decomposition. Specifically, the eigenvalues of Σ are given by $\lambda_1 = (\frac{a}{\sqrt{2}})^2$ and $\lambda_2 = (\frac{b}{\sqrt{2}})^2$, with corresponding eigenvectors $v_1 = (\cos\theta, \sin\theta)'$ and $v_2 = (-\sin\theta, \cos\theta)'$, so that $\Sigma = \lambda_1 v_1 v_1' + \lambda_2 v_2 v_2'$. Note that whereas the Argos reported angle $\theta\epsilon[0°, 180°)$ gives the orientation of the major axis of the ellipse measured from the north going east, in these equations it is measured from east to north. As an example, $a = 1000$ m, $b = 400$ m, and $\theta = 80°$ (Figure 1) leads to $\lambda_1 = (\frac{1000\ m}{\sqrt{2}})^2$, $\lambda_2 = (\frac{400\ m}{\sqrt{2}})^2$, $v_1 = (\cos 10°, \sin 10°)'$, and $v_2 = (-\sin 10°, \cos 10°)'$, and the covariance matrix turns out to be approximately $\begin{bmatrix} 48.6e4 & 7.78e4 \\ 7.78e4 & 5.87e4 \end{bmatrix}$. Conversely, starting with the covariance matrix of the error distribution, eigen analysis can be used to obtain its eigenvalues and eigenvectors and hence $a, b,$ and θ. Moreover, this can be done for any particular choice of the probability P for the ellipse by replacing $\sqrt{2}$ in the equations relating λ_1 and λ_2 to a and b with $\sqrt{q}$, where q is the corresponding percentile of the chi-square distribution on two degrees of freedom, i.e. $\Pr(\chi_2^2 < q) = P$.

The methods used to determine satellite tag pop-up location involve non-linear transformations of Argos estimated geographic location coordinates, such as the Haversine formula. For situations involving relatively small changes in latitude, however, these methods can be well approximated by simple linear transformations of the coordinates, which allows computation of the resulting error distribution based on standard multivariate normal theory. Let $\widehat{P_o} = (\hat{x}_o, \hat{y}_o)'$ denote estimated tag pop-up location, expressed in degrees of longitude and latitude. There are two cases to consider, depending on whether tag drift rate is estimated using two drift locations $P_1 = (x_1, y_1)'$ and $P_2 = (x_2, y_2)'$ of the target tag itself or using two drift locations $P_1^* = (x_1^*, y_1^*)'$ and $P_2^* = (x_2^*, y_2^*)'$ of a suitable proxy tag. In the first case, estimated pop-up location is approximately given by the pair of equations

$$\hat{x}_o = x_1 + \left(-t_1 \frac{x_2 - x_1}{t_2}\right) = \frac{t_1 + t_2}{t_2} x_1 + (-\frac{t_1}{t_2}) x_2$$	(1a)
$$\hat{y}_o = y_1 + \left(-t_1 \frac{y_2 - y_1}{t_2}\right) = \frac{t_1 + t_2}{t_2} y_1 + \left(-\frac{t_1}{t_2}\right) y_2$$	(1b)

where t_1 and t_2 denote drift time between P_o and P_1 and between P_1 and P_2, respectively. In the second case, estimated pop-up location is approximately given by the pair of equations

$$\hat{x}_o = x_1 + \left(-t_1 \frac{x_2^* - x_1^*}{t_2^*}\right) = x_1 + \frac{t_1}{t_2^*} x_1^* + (-\frac{t_1}{t_2^*}) x_2^* \tag{2a}$$

$$\hat{y}_o = y_1 + \left(-t_1 \frac{y_2^* - y_1^*}{t_2^*}\right) = y_1 + \frac{t_1}{t_2^*} y_1^* + \left(-\frac{t_1}{t_2^*}\right) y_2^*, \tag{2b}$$

where t_1 denotes drift time between P_o and P_1, and t_2^* denotes drift time between the two proxy-tag locations P_1^* and P_2^*. Writing $M = \begin{bmatrix} \frac{t_1+t_2}{t_2} & 0 & -\frac{t_1}{t_2} & 0 \\ 0 & \frac{t_1+t_2}{t_2} & 0 & -\frac{t_1}{t_2} \end{bmatrix}$ and $Z = (x_1, y_1, x_2, y_2)'$, equations (1) can be formulated in terms of the matrix equation $(\hat{x}_o, \hat{y}_o)' = MZ$. If we now assume that location errors are bivariate normal and independent between locations, Z will be multivariate normal with covariance matrix $\Gamma = \begin{bmatrix} \Sigma_1 & (0) \\ (0) & \Sigma_2 \end{bmatrix}$ and hence estimated pop-up location $(\hat{x}_o, \hat{y}_o)'$ will have bivariate normal error with covariance matrix given by

$$M\Gamma M' = \begin{bmatrix} \left(\frac{t_1+t_2}{t_2}\right)^2 \sigma_{X_1}^2 + \left(\frac{t_1}{t_2}\right)^2 \sigma_{X_2}^2 & \left(\frac{t_1+t_2}{t_2}\right)^2 \sigma_{X_1Y_1} + \left(\frac{t_1}{t_2}\right)^2 \sigma_{X_2Y_2} \\ \left(\frac{t_1+t_2}{t_2}\right)^2 \sigma_{X_1Y_1} + \left(\frac{t_1}{t_2}\right)^2 \sigma_{X_2Y_2} & \left(\frac{t_1+t_2}{t_2}\right)^2 \sigma_{Y}^2 + \left(\frac{t_1}{t_2}\right)^2 \sigma_{Y_2}^2 \end{bmatrix}. \tag{3}$$

The associated semi-major axis, semi-minor axis, and angle of orientation characterizing the error ellipse can then be obtained as described above (Figure 2). For the second case in which proxy-tag locations are used in estimating pop-up location, a similar formulation of equations (2) in terms of a matrix N applied to the vector of location coordinates $Z = (x_1, y_1, x_1^*, y_1^*, x_2^*, y_2^*)'$ having covariance matrix $\Delta = \begin{bmatrix} \Sigma_1 & (0) & (0) \\ (0) & \Sigma_1^* & (0) \\ (0) & (0) & \Sigma_2^* \end{bmatrix}$ yields the pop-location error covariance matrix

$$N\Delta N' = \begin{bmatrix} \sigma_{X_1}^2 + \left(\frac{t_1}{t_2^*}\right)^2 (\sigma_{X_1^*}^2 + \sigma_{X_2^*}^2) & \sigma_{X_1Y_1} + \frac{t_1}{t_2^*} (\sigma_{X_1^*Y_1^*} + \sigma_{X_2^*Y_2^*}) \\ \sigma_{X_1Y_1} + \frac{t_1}{t_2^*} (\sigma_{X_1^*Y_1^*} + \sigma_{X_2^*Y_2^*}) & \sigma_{Y_1}^2 + \left(\frac{t_1}{t_2^*}\right)^2 (\sigma_{Y_1^*}^2 + \sigma_{Y_2^*}^2) \end{bmatrix}. \tag{4}$$

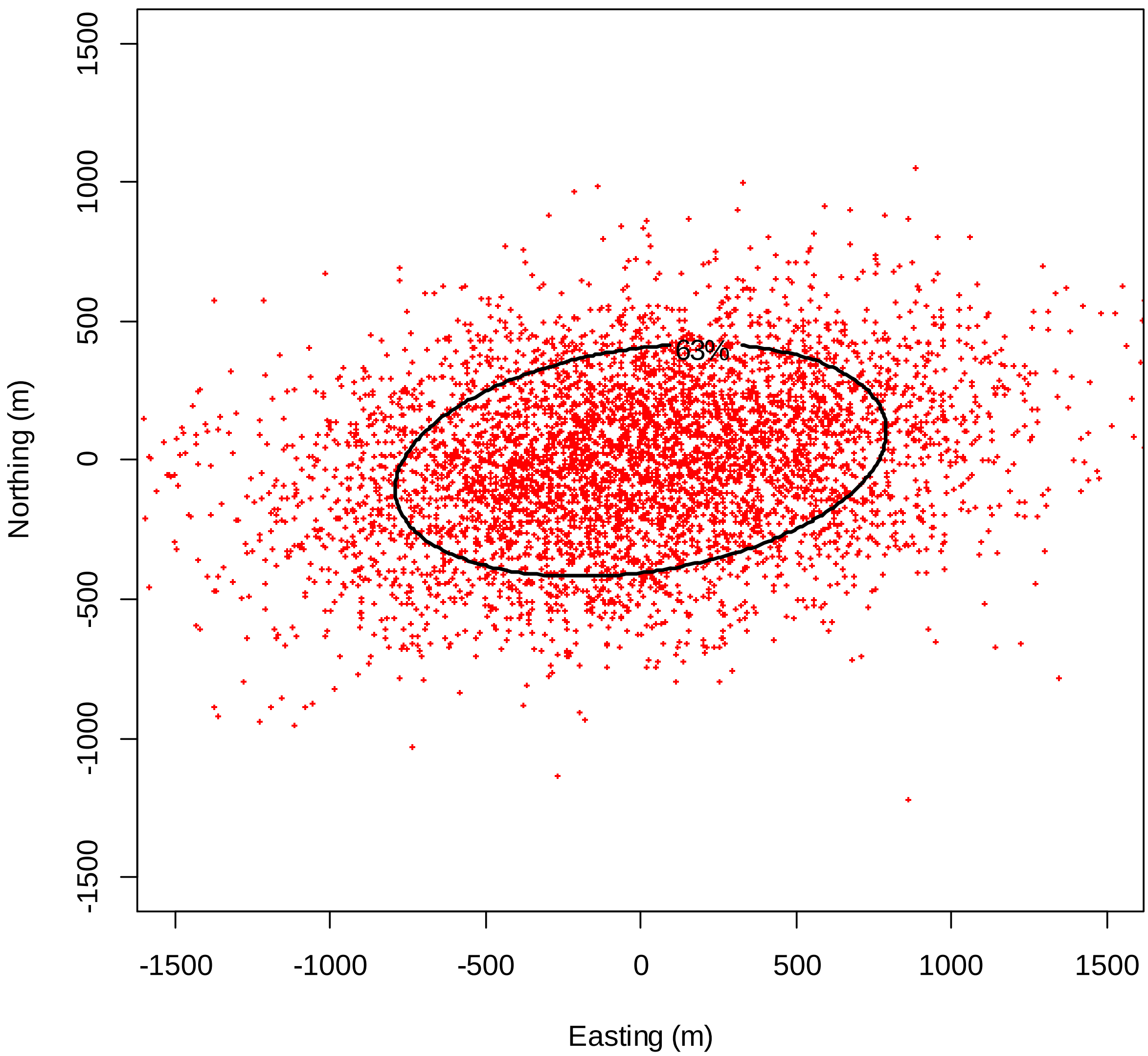

Figure 1. Simulated draws (n = 5000) from a bivariate normal distribution with $\sqrt{2}\sigma$ (63% probability) ellipse having semi-major axis = 1000 m, semi-minor axis = 400 m, and major axis orientation θ = 80º (clockwise from north).

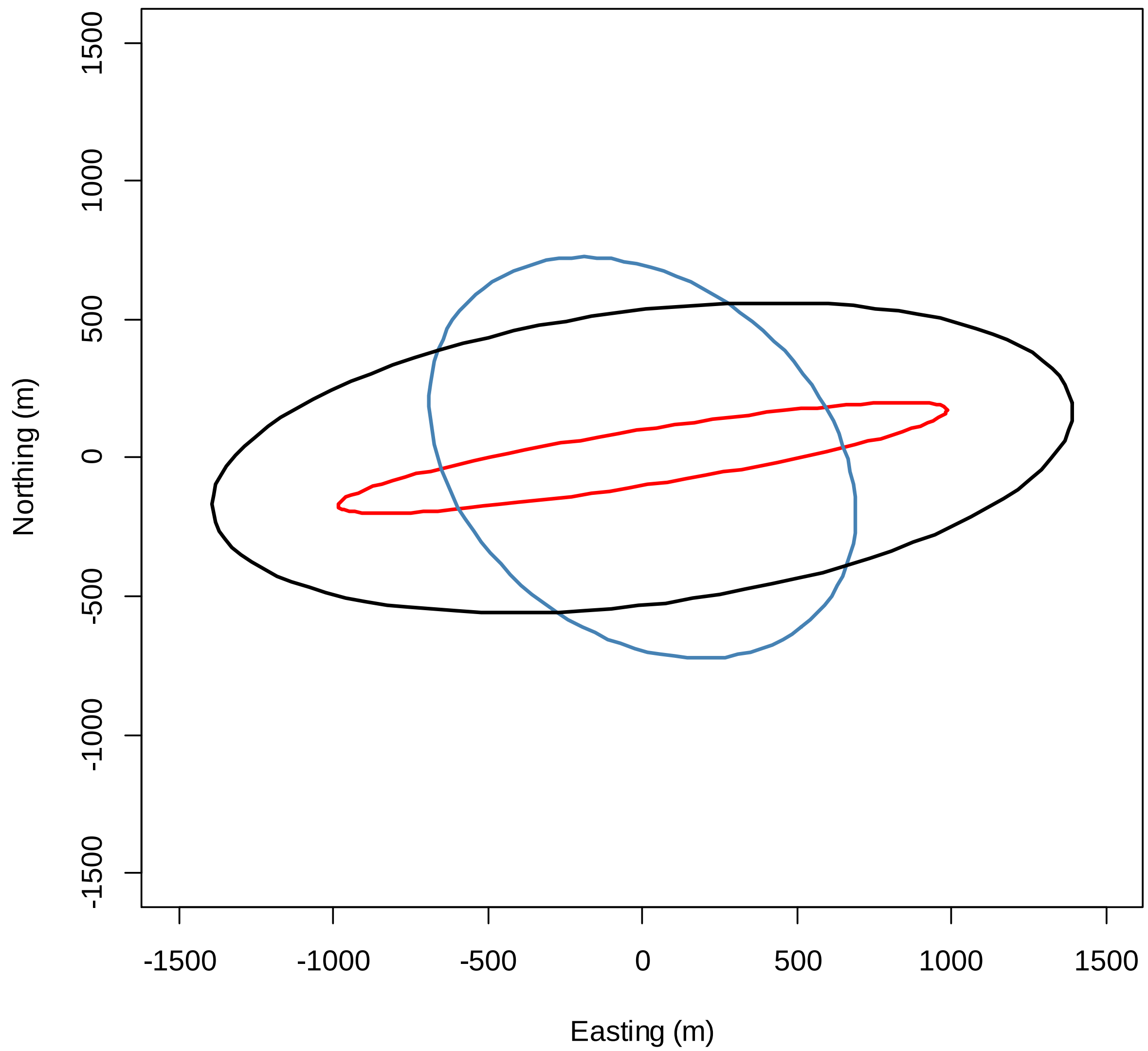

Figure 2. Bivariate normal $\sqrt{2}\sigma$ (63% probability) ellipses for two drift locations P_1 and P_2 (red and blue) and the corresponding estimated pop-up location $\widehat{P_o}$ (black) assuming drift times of 4 h between P_o and P_1 and 6 h between P_1 and P_2. Semi-major axes for P_1, P_2 and $\widehat{P_o}$ are, respectively, 1000 m, 800 m and 1404 m. Error distributions have been centered at the origin to facilitate visual comparison.