

Electronic Supplementary Material (Online Resource 1) for: “Discrete Modeling of Cracking in Reinforced Concrete Structures: Formulation, Size Effect, and Parameter Sensitivity”

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Appendix A: Derivation of cohesive–frictional envelope

This appendix presents the derivation of the cohesive-frictional envelope defining the evolution of concrete strength based on the formulation presented in Section 2 of the main paper. An analysis of the effective model stemming from interactions between progressive cracking (decohesion) modeled using a damage law, frictional contribution modeled using a plasticity framework, and dilatancy modeled through an elastic offset, is warranted. During the derivation, the following equations will be used:

$$\mathbf{t}' = (1 - d_c)K_c \begin{bmatrix} \llbracket u'_{x'} \rrbracket - \llbracket u'_{x'} \rrbracket_{\text{dil}} \\ \frac{\beta^2}{\kappa} \llbracket u'_{y'} \rrbracket \\ \frac{\beta^2}{\kappa} \llbracket u'_{z'} \rrbracket \end{bmatrix} \quad (\text{S1})$$

$$+ d_c K_c \begin{bmatrix} \llbracket u'_{x'} \rrbracket - \llbracket u'_{x'} \rrbracket_{\text{dil}} \\ \llbracket u'_{y'} \rrbracket - \llbracket u'_{y'} \rrbracket_{\mu} \\ \llbracket u'_{z'} \rrbracket - \llbracket u'_{z'} \rrbracket_{\mu} \end{bmatrix} H(-t'_{x'}),$$

$$\alpha_c = \sqrt{\langle \llbracket u'_{x'} \rrbracket \rangle_+^2 + \frac{\beta^2}{\kappa^2} (\llbracket u'_{y'} \rrbracket^2 + \llbracket u'_{z'} \rrbracket^2)}, \quad (\text{S2})$$

$$d_c = \min\left(1 - \frac{q_c}{K_c \alpha_{\max}}, 1\right), \quad (\text{S3})$$

with all terms already explained in the main text. All the equations are written in the local coordinate system of the fracture surface, where x' denotes the direction of the normal to the surface, and y' and z' are the tangents. Further, since the failure surface is sought after, the frictional terms in Eq. S1 can be set as

$$K(\llbracket u'_{y'} \rrbracket - \llbracket u'_{y'} \rrbracket_{\mu}) = \mu |t'_{x'}| \frac{t'_{y'}}{\|\mathbf{t}_T\|} \quad (\text{S4})$$

and

$$K(\llbracket u'_{z'} \rrbracket - \llbracket u'_{z'} \rrbracket_{\mu}) = \mu |t'_{x'}| \frac{t'_{z'}}{\|\mathbf{t}_T\|}, \quad (\text{S5})$$

where $\|\mathbf{t}_T\| = \sqrt{t'^2_{y'} + t'^2_{z'}}$.

From Eq. S1, the crack openings can be expressed in terms of remaining variables as

$$\langle \llbracket u'_{x'} \rrbracket \rangle_+ = \frac{\langle t'_{x'} \rangle_+}{(1 - d_c + H(-t'_{x'})d_c)K_c} + \eta \sqrt{\llbracket u'_{y'} \rrbracket^2 + \llbracket u'_{z'} \rrbracket^2} \quad (\text{S6})$$

$$\llbracket u'_{y'} \rrbracket = \frac{\kappa}{\beta^2} \frac{t'_{y'} - d_c \mu |t'_{x'}| \frac{t'_{y'}}{\|\mathbf{t}_T\|} H(-t'_{x'})}{(1 - d_c)K_c} \quad (\text{S7})$$

$$\llbracket u'_{z'} \rrbracket = \frac{\kappa}{\beta^2} \frac{t'_{z'} - d_c \mu |t'_{x'}| \frac{t'_{z'}}{\|\mathbf{t}_T\|} H(-t'_{x'})}{(1 - d_c) K_c} \quad (\text{S8})$$

Substituting expressions for $\llbracket u'_{y'} \rrbracket$ and $\llbracket u'_{z'} \rrbracket$ into $\sqrt{\llbracket u'_{y'} \rrbracket^2 + \llbracket u'_{z'} \rrbracket^2}$, we obtain:

$$\sqrt{\llbracket u'_{y'} \rrbracket^2 + \llbracket u'_{z'} \rrbracket^2} = \frac{\kappa}{\beta^2 K_c (1 - d_c)} \sqrt{(\|\mathbf{t}_T\| - d_c \mu |t'_{x'}| H(-t'_{x'}))^2} \quad (\text{S9})$$

Then,

$$\begin{aligned} \langle \llbracket u'_{x'} \rrbracket \rangle_+ &= \frac{\langle t'_{x'} \rangle_+}{(1 - d_c + H(-t'_{x'}) d_c) K_c} \\ &+ \frac{\eta \kappa}{\beta^2 K_c (1 - d_c)} \sqrt{(\|\mathbf{t}_T\| - d_c \mu |t'_{x'}| H(-t'_{x'}))^2} \end{aligned} \quad (\text{S10})$$

Analyzing Eq. S10, it can be noted that the term $\frac{\langle t'_{x'} \rangle_+}{1 - d_c + H(-t'_{x'}) d_c}$ vanishes for $t'_{x'} \leq 0$, while for $t'_{x'} > 0$ it evaluates to $\frac{\langle t'_{x'} \rangle_+}{1 - d_c}$. The term $|t'_{x'}| H(-t'_{x'})$ is equivalent to $\langle -t'_{x'} \rangle_+$.

Substituting above expressions into Eq. S2, multiplying both sides by $K_c(1 - d_c)$, and squaring, the following expression is obtained:

$$\begin{aligned} q_c^2 &= \left(\langle t'_{x'} \rangle_+ + \frac{\eta \kappa}{\beta^2} \sqrt{(\|\mathbf{t}_T\| - d_c \mu \langle -t'_{x'} \rangle_+)^2} \right)^2 \\ &+ \frac{1}{\beta^2} (\|\mathbf{t}_T\| - d_c \mu \langle -t'_{x'} \rangle_+)^2. \end{aligned} \quad (\text{S11})$$

The envelope of the function defined by Eq. S11 takes the following form:

$$F = \sqrt{\frac{\|\mathbf{t}_T\|^2 + \frac{\beta^2 \langle t'_{x'} \rangle_+}{\eta^2 \kappa^2 + \beta^2} (\beta^2 \langle t'_{x'} \rangle_+ + 2\eta \kappa \|\mathbf{t}_T\|)}{\left(\frac{q_c \beta^2}{\sqrt{\eta^2 \kappa^2 + \beta^2}} + d_c \mu \langle -t'_{x'} \rangle_+ \right)^2}} - 1 \stackrel{!}{=} 0 \quad (\text{S12})$$

Figure S1 shows a three dimensional representation of the derived failure surface (Eq. S12) evaluated for parameters used in this paper.

Appendix B: Bond element formulation

This Appendix presents the finite element formulation of the reinforcement and coupling elements. The section introduces the discretization strategy,

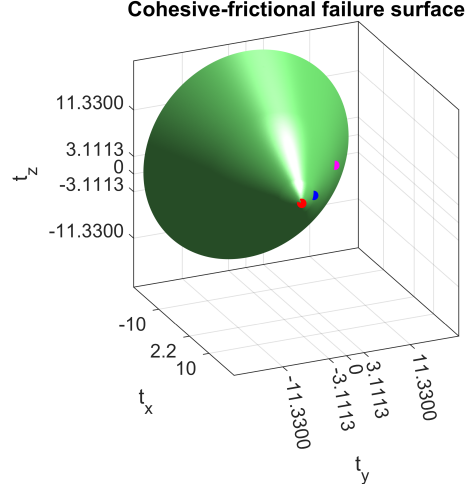


Fig. S1: 3D view of cohesive frictional failure surface with dilatancy using the parameters from Table 2 of the main paper. These parameters result in tensile strength $f_t = 2.2$ MPa (red point), shear strength of $\frac{\beta^2}{\sqrt{\eta^2 \kappa^2 + \beta^2}} f_t = 3.1113$ MPa (blue point), and shear frictional strength of $\frac{\beta^2}{\sqrt{\eta^2 \kappa^2 + \beta^2}} f_t + d_c \mu |t'_{x'}| = 11.33$ MPa for $t'_{x'} = -10$ MPa (magenta point).

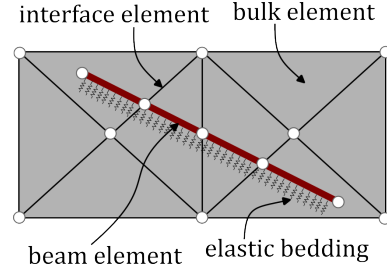


Fig. S2: Penalty-based coupling between rebar and the background mesh - model components and discretization.

the approximation of geometry and displacement field, the integration procedure, as well as the derivation of the internal force vector and the stiffness matrix from the element contribution to the virtual work of internal forces facilitating the coupling $\delta W_b^{\text{coh},e}$.

The coupling element formulation depends on certain conditions regarding the placement of reinforcement line elements in the finite element mesh. The assumptions regarding the pre-processing are:

- Each line representing a rebar is split into line segments (Timoshenko beam elements) so that each segment lies strictly within one solid element's volume. The solid element is denoted as a *parent* element for that line segment.
- One solid element may act as a parent element for multiple line segments (fully enclosing them). However, a line segment is not allowed to span multiple solid elements.

Figure S2 illustrates the discretization strategy. The pre-processing is performed before the simulation starts by identifying the intersections between lines that define reinforcement and the solid elements, and subsequently discretizing the lines into an appropriate number of Timoshenko beam elements. More details can be found in [1].

The finite element geometry of the coupling element consists of a union of the reinforcement and solid element geometries. In this regard, the interface between the reinforcement and solid is not explicitly resolved, but rather, the reinforcement is "immersed" into the background finite element mesh. This implementation detail enables easy access to nodal displacements, internal states, and functionalities of both elements. The finite element representations and node numbering of the solid, reinforcement, and coupling elements are illustrated in Figure S3a, S3b, and S3c. In this work, a standard linear tetrahedral element is used to discretize the solid phase, and a linear Timoshenko beam element with selective reduced integration is utilized to discretize the reinforcement phase. At the level of Timoshenko beam element integration point, a cross section object containing 9 integration points is introduced to accurately capture the stress states resulting from partial plastification of the cross section (see Table S1 and Figure S3d). Detailed formulation can be found in [1]. The actual coupling takes place along the interface between reinforcement and solid elements indicated by the line 1 – 2 in the figure.

The geometry of the coupling element over which the integration is performed is a line coinciding with the Timoshenko beam element and has a diameter ϕ which is the same as the diameter of reinforcement. Therefore, the geometry and the JACOBIAN of the coupling are defined by the expressions analogous to those for the Timoshenko

beam element, defined as $|J| = \frac{L^e}{2}$, where L^e is the length of the line segment.

As mentioned above, the coupling element has a full access to solid element and beam element data, including nodal coordinates and displacement values, so the coupling element can compute a rotation matrix $\mathbf{R}$ which transforms quantities, such as nodal displacements, to align with the local coordinate system of the Timoshenko beam element. Next, utilizing the shape functions ($\mathbf{N}(\boldsymbol{\xi})$) and the nodal displacements ($\mathbf{u}^e$) of the matrix and reinforcement elements, the relative slip and virtual relative slip in the local coordinate system of the bond, which coincides with the local coordinate system of the Timoshenko beam element, is calculated as follows:

$$\begin{aligned} \llbracket \mathbf{u} \rrbracket'_b(\xi_{1,r}) &= \mathbf{R}(\mathbf{u}_r(\xi_{1,r}) - \mathbf{u}_s(\boldsymbol{\xi}_s)) \\ &= \mathbf{R} \underbrace{[\mathbf{N}_r \quad -\mathbf{N}_s]}_{N_u} \underbrace{\begin{bmatrix} \hat{\mathbf{u}}_r^e \\ \mathbf{u}_s^e \end{bmatrix}}_{\mathbf{u}_b^e}, \end{aligned} \quad (\text{S13})$$

$$\begin{aligned} \delta \llbracket \mathbf{u} \rrbracket'_b(\xi_{1,r}) &= \mathbf{R}(\delta \mathbf{u}_r(\xi_{1,r}) - \delta \mathbf{u}_s(\boldsymbol{\xi}_s)) \\ &= \mathbf{R} \underbrace{[\mathbf{N}_r \quad -\mathbf{N}_s]}_{N_u} \underbrace{\begin{bmatrix} \delta \hat{\mathbf{u}}_r^e \\ \delta \mathbf{u}_s^e \end{bmatrix}}_{\delta \mathbf{u}_b^e}, \end{aligned} \quad (\text{S14})$$

where $\mathbf{u}_r(\xi_{1,r}) = \mathbf{N}_r(\boldsymbol{\xi})\hat{\mathbf{u}}_r^e$ and $\mathbf{u}_s(\boldsymbol{\xi}) = \mathbf{N}_s(\boldsymbol{\xi})\mathbf{u}_s^e$ are the displacement vectors of the reinforcement and the matrix element evaluated using shape functions ($\mathbf{N}_r$ and $\mathbf{N}_s$) at isoparametric coordinates $\xi_{1,r}$, $\boldsymbol{\xi}_s$, and N_u denotes the operator used for calculating bond detachment.

It is worth noting that the isoparametric coordinate $\boldsymbol{\xi}$ has a different meaning in the context of solid and beam elements (see Figure S3c). Namely, for the 3D solid tetrahedral element it is a vector $\boldsymbol{\xi}_s = [\xi_1 \ \xi_2 \ \xi_3]^T$, defined in the coordinate system of the parent solid element, whereas for the beam element it is actually a scalar $\xi_{1,r}$ defined in the coordinate system of the parent beam element.

The quantity that is common for both of them is the physical coordinate of the bond integration point $\mathbf{X}^{ip}$, with ip denoting the integration point index at which the slip is sought after. Since the local coordinates (ξ_1) of the integration points of the bond are predefined (Table S2), they are used as a starting point for determining $\boldsymbol{\xi}_s$ as follows. For each integration point isoparametric coordinate $\xi_{1,r}^{ip}$, the physical coordinate of the

integration point that lies on the beam element is calculated utilizing shape functions ($\mathbf{N}_r$) and beam end-node coordinates ($\mathbf{X}_r^e$). After the physical coordinates of the integration point are found, an iterative procedure is performed inside of the parent solid element to determine the isoparametric coordinates ξ_s corresponding to the physical coordinates $\mathbf{X}^{ip}$. The procedure summarized here is given in more detail in Algorithm 1.

Algorithm 1 Compute local coordinates of the solid element (ξ_s), given the isoparametric coordinates of the beam element ($\xi_{1,r}$), and the physical coordinates of the beam and the solid element nodes ($\mathbf{X}_r^e, \mathbf{X}_s^e$).

Require: $\xi_{1,r}, \mathbf{X}_r^e, \mathbf{X}_s^e$

Ensure: ξ_s

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 $\mathbf{X}_r \leftarrow \mathbf{X}_r(\xi_{1,r})$             $\triangleright \mathbf{X}(\xi_1) = \mathbf{N}_r(\xi_1)\mathbf{X}_r^e$ 
 $\xi_s = 0, \Delta\xi_s = \mathbf{I}$             $\triangleright$  Initialization
while  $\|\Delta\xi_s\| > tol$  do
     $\mathbf{X}_s \leftarrow \mathbf{X}_s(\xi_s)$         $\triangleright \mathbf{X}_s(\xi_s) = \mathbf{N}_s(\xi_s)\mathbf{X}_s^e$ 
     $\Delta\mathbf{X} \leftarrow \mathbf{X}_r - \mathbf{X}_s$ 
     $\mathbf{J} \leftarrow \mathbf{J}(\xi_s)$           $\triangleright \mathbf{J}(\xi_s) = \frac{\partial \mathbf{N}_s}{\partial \xi_s}(\xi_s)\mathbf{X}_s^e$ 
     $\Delta\xi_s \leftarrow \mathbf{J}^{-1}\Delta\mathbf{X}$ 
     $\xi_s \leftarrow \xi_s + \Delta\xi_s$ 
end while

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In standard linear Timoshenko beam finite elements, the displacement and rotation degrees of freedom are uncoupled, and so the coupling is enforced only for the displacement degrees of freedom. The component of $[\mathbf{u}]'_b$ in the x' direction represents the slip between the fiber and the matrix.

Assuming all quantities at time step $n + 1$ are known ($\mathbf{t}_{b',n+1}$, $[\mathbf{u}]'_{b,n+1}$, the subscripts $n + 1$ will be omitted for simplicity), the internal virtual work integral over the interface between reinforcement and concrete (Γ_b^e) can be evaluated in VOIGT notation as:

$$\begin{aligned}
\delta W_b^{\text{coh},e} &= \iint_{\Gamma_b^e} \delta [\mathbf{u}]'_b{}^T \mathbf{t}'_b dA = \int_{L^e} \delta [\mathbf{u}]'_b{}^T \mathbf{t}'_b \pi \phi dx' \\
&= \int_{-1}^1 (\mathbf{R} \mathbf{N}_u \delta \mathbf{u}_b^e)^T \mathbf{t}'_b |J| \pi \phi d\xi'_1 \\
&= \delta \mathbf{u}_b^{e,T} \int_{-1}^1 \mathbf{N}_u^T \mathbf{R}^T \mathbf{t}'_b |J| \pi \phi d\xi'_1 = \delta \mathbf{u}_b^{e,T} \mathbf{f}_{\text{int}}^e,
\end{aligned} \tag{S15}$$

where γ_b^e is surface area of the interface between the reinforcement and the matrix accounting for loss of contact due to slip, L^e is the length of the reinforcement segment within the solid element, ϕ is the diameter of the reinforcement, $|J| = \frac{L^e}{2}$ is the JACOBIAN, and $\mathbf{t}'_b$ is the traction vector of the bond.

The coupling element stiffness matrix is obtained as a variation of the element internal force vector with respect to the nodal displacements as:

$$\begin{aligned}
\mathbf{K}^e &= \frac{\partial \mathbf{f}_{\text{int}}^e}{\partial \mathbf{u}_b^e} = \int_{-1}^1 \mathbf{N}_u^T \mathbf{R}^T \frac{\partial \mathbf{t}'_b}{\partial \mathbf{u}_b^e} \frac{[\mathbf{u}]'_b}{[\mathbf{u}]'_b} \frac{[\mathbf{u}]_b}{\partial \mathbf{u}_b^e} |J| \pi \phi d\xi'_1 \\
&= \int_{-1}^1 \mathbf{N}_u^T \mathbf{R}^T \mathbb{C}'_{b,\text{alg}} \mathbf{R} \mathbf{N}_u |J| \pi \phi d\xi'_1,
\end{aligned} \tag{S16}$$

where $\mathbb{C}'_{b,\text{alg}}$ is the algorithmic tangent matrix is evaluated using the Complex-Step Derivative Approximation (CSDA) procedure [1, 2].

#	ξ_2'	ξ_3'	Weight
1	1.0	0.0	$A/24.0$
2	-1.0	0.0	$A/24.0$
3	0.0	1.0	$A/24.0$
4	0.0	-1.0	$A/24.0$
5	0.5	0.5	$A/6.0$
6	0.5	-0.5	$A/6.0$
7	-0.5	-0.5	$A/6.0$
8	-0.5	0.5	$A/6.0$
9	0.0	0.0	$A/6.0$

Table S1: Integration rule by Albrecht and Collatz [3] for evaluation of 2-dimensional integrals over a circular surface. The integration rule is defined for a bi-unit circle, so the physical coordinates need to be calculated as $y' = \xi_2 \frac{d}{2}$ and $z' = \xi_3 \frac{d}{2}$, where d is the cross-section diameter. A denotes the physical area of the cross-section.

The internal force and stiffness matrix integrals defined by Eqs. S15 and S16 are evaluated using a 6-point 1D Gaussian integration rule [4] along the line coinciding with the middle line of the reinforcement. Table S2 provides the integration weights and coordinates in the isoparametric space of the line element.

#	ξ_1'	Weight
1	-0.9324695142031521	0.1713244923791704
2	-0.6612093864662645	0.3607615730481386
3	-0.2386191860831969	0.4679139345726910
4	0.2386191860831969	0.4679139345726910
5	0.6612093864662645	0.3607615730481386
6	0.9324695142031521	0.1713244923791704

Table S2: 1D GAUSSIAN integration rule [4] along reinforcement direction utilized for integration of coupling element.

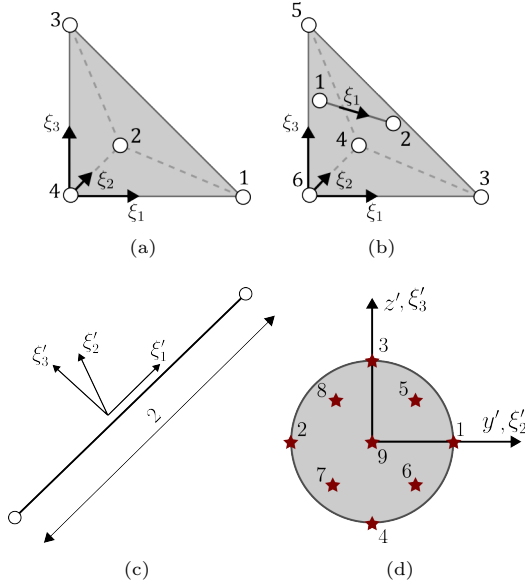


Fig. S3: Finite element geometries and node numbering of the (a) solid, (b) coupling elements, (c) reinforcement, and (d) enumeration and locations of integration points on the circular cross-section of the reinforcement element

Appendix C: Implicit–Explicit integration scheme

One of the crucial aspects that determines the success of any computational model is its robustness, which refers to the model’s capability to consistently produce reliable results across a wide range of challenging problems. The challenges related to robustness, notably when modeling crack formation/propagation using finite element methods, have been extensively documented in the literature ([5–8]). The robustness issues are prevalent

whenever softening constitutive models are utilized. This section briefly summarizes the fundamental problem and then describes an approach utilized in this work to tackle this issue.

After spatial and temporal discretization ($t \in [0, T]$) of the nonlinear solid mechanics problem, the discrete balance of linear momentum equations (neglecting the inertia terms and assuming conservative external loads) at time t_{n+1} are expressed as follows:

$$\mathbf{r}(\mathbf{u}_{n+1}, t_{n+1}) = \mathbf{f}_{\text{int}}(\mathbf{u}_{n+1}, t_{n+1}) - \mathbf{f}_{\text{ext}}(t_{n+1}) \stackrel{!}{=} \mathbf{0}, \quad (\text{S17})$$

where $\mathbf{r}(\mathbf{u}_{n+1}, t_{n+1})$ is the residual or out-of-balance force vector corresponding to the non-fixed degrees of freedom, which needs to be nullified by finding the solution $\mathbf{u}_{n+1}$ (nodal displacements vector) of the nonlinear system of equations. This is typically done utilizing the NEWTON-RAPHSON iterative procedure on a linearized system of equations. The structural stiffness matrix can be strongly affected by the loss of ellipticity of the underlying differential equation once the material failure occurs (loss of material stability in the sense of Drucker [9]). The progressive failure manifests itself through reduction of eigenvalues of the structural stiffness matrix, and a consequent loss of its positive-definiteness. This poses tremendous problems for the NEWTON-RAPHSON method, and finally, the procedure of finding the solution fails.

In this work, a simple yet powerful approach is utilized to circumvent these issues. The essence of the IMPL-EX integration scheme, proposed by Oliver et al. [5], is based on a combination of explicit constitutive law integration involving extrapolation of the internal variables and the implicit correction once the equilibrium has been achieved. This approach was chosen due to its favorable properties, such as maintaining symmetry and positive definiteness of the resulting matrices.

In the following, the IMPL-EX scheme will be briefly explained in the context of fracture model used in this work, however, it can be used in the context of any nonlinear constitutive relation.

The IMPL-EX integration of cohesive-frictional traction-separation relations described in Section 2 of the main paper is based on extrapolating the internal variables ($\check{\mathbf{m}}_{n+1}$) at the time

step t_{n+1} by leveraging its implicitly obtained values of internal variables from the earlier two time steps, $(\bullet_n)$ and $(\bullet_{n-1})$, also accounting for possible varying time step size. The internal variables chosen for extrapolation in this work are the crack opening-like internal variable $(\tilde{\alpha}_{c,n+1})$ used for evaluation of the damage progression, the discrete frictional slip multiplier $(\Delta\tilde{\lambda}_{\mu,n+1})$ used for calculation of frictional contribution to traction vector, and the tangential crack opening is also treated as an internal variable $(\tilde{u}_{T,n+1} = \sqrt{\llbracket \tilde{u}'_{y'} \rrbracket_{n+1}^2 + \llbracket \tilde{u}'_{z'} \rrbracket_{n+1}^2})$ only for the evaluation crack opening due to dilatancy (Eq. ??). The extrapolated internal variables are calculated as follows:

$$\begin{aligned}\tilde{\alpha}_{c,n+1} &= \alpha_{c,n} + \frac{\Delta t_{n+1}}{\Delta t_n}(\alpha_{c,n} - \alpha_{c,n-1}), \\ \Delta\tilde{\lambda}_{\mu,n+1} &= \frac{\Delta t_{n+1}}{\Delta t_n}\Delta\lambda_{\mu,n}, \\ \tilde{u}_{T,n+1} &= u_{T,n} + \frac{\Delta t_{n+1}}{\Delta t_n}(u_{T,n} - u_{T,n-1}),\end{aligned}\tag{S18}$$

where Δt_{n+1} and Δt_n are the previous time increments. The IMPL-EX traction $\tilde{\mathbf{t}}'_{n+1}$ is evaluated by substituting the extrapolated internal variables into equations introduced in Section 2 of the main paper, resulting in the discrete form of Eq. S1:

$$\begin{aligned}\tilde{\mathbf{t}}'_{n+1} &= (1 - d_c(\tilde{\alpha}_{c,n+1}))K_c \begin{bmatrix} \llbracket u'_{x'} \rrbracket - \llbracket \tilde{u}'_{x'} \rrbracket_{\text{dil}}(\tilde{u}_{T,n+1}) \\ \frac{\beta^2}{\kappa} \llbracket u'_{y'} \rrbracket \\ \frac{\beta^2}{\kappa} \llbracket u'_{z'} \rrbracket \end{bmatrix} \\ &+ d_c(\tilde{\alpha}_{c,n+1})K_c \begin{bmatrix} \llbracket u'_{x'} \rrbracket - \llbracket \tilde{u}'_{x'} \rrbracket_{\text{dil}}(\tilde{u}_{T,n+1}) \\ \llbracket u'_{y'} \rrbracket - \llbracket \tilde{u}'_{y'} \rrbracket_{\mu}(\Delta\tilde{\lambda}_{\mu,n+1}) \\ \llbracket u'_{z'} \rrbracket - \llbracket \tilde{u}'_{z'} \rrbracket_{\mu}(\Delta\tilde{\lambda}_{\mu,n+1}) \end{bmatrix} H(-\tilde{t}'_{x'}).\end{aligned}\tag{S19}$$

The value of the contact activation function $H(-\tilde{t}'_{x'}) = H(\llbracket u'_{x'} \rrbracket - \llbracket \tilde{u}'_{x'} \rrbracket_{\text{dil}}(\tilde{u}_{T,n+1}))$ is determined utilizing the active-set strategy described in [1]. The IMPL-EX material tangent $\tilde{\mathbf{C}}'_{n+1,\text{alg}} = \frac{\partial \tilde{\mathbf{t}}'_{n+1}}{\partial \mathbf{u}'}$ is evaluated using the Complex-Step Derivative Approximation (CSDA) procedure [1, 2]. It should be noted that the extrapolation of $\tilde{u}_{T,n+1}$ leads to preserving the symmetry of the algorithmic tangent matrix.

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