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Phase-locking in nonidentical coupled light-fueled liquid crystal elastomer self-oscillators

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Section S1: Derivations of Eq. (8)

The dimensionless forms of Eqs. (5) and (7) are

$$\begin{cases} \bar{F}_{f1}(\bar{t}) = \bar{K}_{f1} \left[\bar{w}_1(\bar{t}) - \bar{C}_0 \int_0^1 \bar{T}(\bar{X}_1, \bar{t}) d\bar{X}_1 \right] \\ \bar{F}_{f2}(\bar{t}) = \bar{K}_{f2} \left[\bar{w}_2(\bar{t}) - \bar{C}_0 \int_0^1 \bar{T}(\bar{X}_2, \bar{t}) d\bar{X}_2 \right] \end{cases} \quad (S1)$$

$$\begin{cases} \bar{\tau} \frac{d\bar{T}(\bar{X}_1, \bar{t})}{d\bar{t}} = \bar{b} \left\{ \bar{X}_1 \left[\bar{w}_1(\bar{t}) - \bar{C}_0 \int_0^1 \bar{T}(\bar{X}_1, \bar{t}) d\bar{X}_1 \right] + \bar{C}_0 \int_0^{\bar{X}_1} \bar{T}(\bar{X}_1, \bar{t}) d\bar{X}_1 + \bar{X}_1 \right\} + \bar{Q} \\ \quad - \bar{T}(\bar{X}_1, \bar{t}) \\ \bar{\tau} \frac{d\bar{T}(\bar{X}_2, \bar{t})}{d\bar{t}} = \bar{b} \left\{ \bar{X}_2 \left[\bar{w}_2(\bar{t}) - \bar{C}_0 \int_0^1 \bar{T}(\bar{X}_2, \bar{t}) d\bar{X}_2 \right] + \bar{C}_0 \int_0^{\bar{X}_2} \bar{T}(\bar{X}_2, \bar{t}) d\bar{X}_2 + \bar{X}_2 \right\} + \bar{Q} \\ \quad - \bar{T}(\bar{X}_2, \bar{t}) \end{cases} \quad (S2)$$

For $\bar{\tau} \ll 1$, the temperature distribution can be expanded as $\bar{T}(\bar{X}_i, \bar{t}) = \bar{T}^{(0)}(\bar{X}_i, \bar{t}) + \bar{\tau} \bar{T}^{(1)}(\bar{X}_i, \bar{t}) + O(\bar{\tau}^2)$ (for $i = 1, 2$). Substituting this expansion into Eq. (S2) and then equating the coefficients of the zeroth-order and first-order terms in $\bar{\tau}$ yields the following equations

$$\begin{cases} \bar{T}^{(0)}(\bar{X}_1, \bar{t}) = \bar{b} \left\{ \bar{X}_1 \left[\bar{w}_1(\bar{t}) - \bar{C}_0 \int_0^1 \bar{T}^{(0)}(\bar{X}_1, \bar{t}) d\bar{X}_1 \right] + \bar{C}_0 \int_0^{\bar{X}_1} \bar{T}^{(0)}(\bar{X}_1, \bar{t}) d\bar{X}_1 + \bar{X}_1 \right\} + \bar{Q} \\ \bar{T}^{(0)}(\bar{X}_2, \bar{t}) = \bar{b} \left\{ \bar{X}_2 \left[\bar{w}_2(\bar{t}) - \bar{C}_0 \int_0^1 \bar{T}^{(0)}(\bar{X}_2, \bar{t}) d\bar{X}_2 \right] + \bar{C}_0 \int_0^{\bar{X}_2} \bar{T}^{(0)}(\bar{X}_2, \bar{t}) d\bar{X}_2 + \bar{X}_2 \right\} + \bar{Q} \end{cases} \quad (S3)$$

$$\begin{cases} \frac{d\bar{T}^{(1)}(\bar{X}_1, \bar{t})}{d\bar{t}} = \bar{b} \bar{C}_0 \left[-\bar{X}_1 \int_0^1 \bar{T}^{(1)}(\bar{X}_1, \bar{t}) d\bar{X}_1 + \int_0^{\bar{X}_1} \bar{T}^{(1)}(\bar{X}_1, \bar{t}) d\bar{X}_1 \right] - \bar{T}^{(1)}(\bar{X}_1, \bar{t}) \\ \frac{d\bar{T}^{(1)}(\bar{X}_2, \bar{t})}{d\bar{t}} = \bar{b} \bar{C}_0 \left[-\bar{X}_2 \int_0^1 \bar{T}^{(1)}(\bar{X}_2, \bar{t}) d\bar{X}_2 + \int_0^{\bar{X}_2} \bar{T}^{(1)}(\bar{X}_2, \bar{t}) d\bar{X}_2 \right] - \bar{T}^{(1)}(\bar{X}_2, \bar{t}) \end{cases} \quad (S4)$$

Considering $\bar{T}^{(0)}(\bar{X}_i=0, \bar{t}) = \bar{Q}$ and $\bar{T}^{(0)}(\bar{X}_i=1, \bar{t}) = \bar{b}[\bar{w}_i(\bar{t})+1] + \bar{Q}$, solving Eq. (S3)

yields

$$\begin{cases} \bar{T}^{(0)}(\bar{X}_1, \bar{t}) = \frac{\bar{b}[\bar{w}_1(\bar{t})+1]}{e^{\bar{C}_0\bar{b}} - 1} (e^{\bar{C}_0\bar{b}\bar{X}_1} - 1) + \bar{Q} \\ \bar{T}^{(0)}(\bar{X}_2, \bar{t}) = \frac{\bar{b}[\bar{w}_2(\bar{t})+1]}{e^{\bar{C}_0\bar{b}} - 1} (e^{\bar{C}_0\bar{b}\bar{X}_2} - 1) + \bar{Q} \end{cases}. \quad (S5)$$

Then, substituting Eq. (S5) into Eq. (S4) gives

$$\begin{cases} \bar{T}^{(1)}(\bar{X}_1, \bar{t}) = \frac{\bar{b}\bar{w}_1(\bar{t})}{e^{\bar{C}_0\bar{b}} - 1} \left[\frac{(e^{\bar{C}_0\bar{b}\bar{X}_1} - 1)(\bar{C}_0\bar{b}e^{\bar{C}_0\bar{b}} - e^{\bar{C}_0\bar{b}} + 1)}{e^{\bar{C}_0\bar{b}} - 1} - \bar{C}_0\bar{b}\bar{X}_1e^{\bar{C}_0\bar{b}\bar{X}_1} \right] \\ \bar{T}^{(1)}(\bar{X}_2, \bar{t}) = \frac{\bar{b}\bar{w}_2(\bar{t})}{e^{\bar{C}_0\bar{b}} - 1} \left[\frac{(e^{\bar{C}_0\bar{b}\bar{X}_2} - 1)(\bar{C}_0\bar{b}e^{\bar{C}_0\bar{b}} - e^{\bar{C}_0\bar{b}} + 1)}{e^{\bar{C}_0\bar{b}} - 1} - \bar{C}_0\bar{b}\bar{X}_2e^{\bar{C}_0\bar{b}\bar{X}_2} \right] \end{cases}. \quad (S6)$$

Finally, the asymptotic form of the temperature field is

$$\begin{cases} \bar{T}(\bar{X}_1, \bar{t}) = \frac{\bar{b}[\bar{w}_1(\bar{t})+1]}{e^{\bar{C}_0\bar{b}} - 1} (e^{\bar{C}_0\bar{b}\bar{X}_1} - 1) + \bar{Q} \\ \quad + \bar{\tau} \frac{\bar{b}\bar{w}_1(\bar{t})}{e^{\bar{C}_0\bar{b}} - 1} \left[\frac{(e^{\bar{C}_0\bar{b}\bar{X}_1} - 1)(\bar{C}_0\bar{b}e^{\bar{C}_0\bar{b}} - e^{\bar{C}_0\bar{b}} + 1)}{e^{\bar{C}_0\bar{b}} - 1} - \bar{C}_0\bar{b}\bar{X}_1e^{\bar{C}_0\bar{b}\bar{X}_1} \right] \\ \bar{T}(\bar{X}_2, \bar{t}) = \frac{\bar{b}[\bar{w}_2(\bar{t})+1]}{e^{\bar{C}_0\bar{b}} - 1} (e^{\bar{C}_0\bar{b}\bar{X}_2} - 1) + \bar{Q} \\ \quad + \bar{\tau} \frac{\bar{b}\bar{w}_2(\bar{t})}{e^{\bar{C}_0\bar{b}} - 1} \left[\frac{(e^{\bar{C}_0\bar{b}\bar{X}_2} - 1)(\bar{C}_0\bar{b}e^{\bar{C}_0\bar{b}} - e^{\bar{C}_0\bar{b}} + 1)}{e^{\bar{C}_0\bar{b}} - 1} - \bar{C}_0\bar{b}\bar{X}_2e^{\bar{C}_0\bar{b}\bar{X}_2} \right] \end{cases}. \quad (S7)$$

By substituting Eq. (S7) into Eq. (S1), the fiber tensions are expressed as

$$\begin{cases} \bar{F}_{f1}(\bar{t}) = \frac{\bar{K}_{f1}\bar{C}_0\bar{b}}{e^{\bar{C}_0\bar{b}} - 1} \bar{w}_1(\bar{t}) + \bar{K}_{f1}\bar{C}_0\bar{b}\bar{\tau} \frac{1 - e^{\bar{C}_0\bar{b}} + \bar{C}_0\bar{b}e^{\bar{C}_0\bar{b}}}{(e^{\bar{C}_0\bar{b}} - 1)^2} \bar{w}_1(\bar{t}) \\ \quad + \bar{K}_{f1} \left(\frac{\bar{C}_0\bar{b}}{e^{\bar{C}_0\bar{b}} - 1} - 1 - \bar{C}_0\bar{Q} \right) \\ \bar{F}_{f2}(\bar{t}) = \frac{\bar{K}_{f2}\bar{C}_0\bar{b}}{e^{\bar{C}_0\bar{b}} - 1} \bar{w}_2(\bar{t}) + \bar{K}_{f2}\bar{C}_0\bar{b}\bar{\tau} \frac{1 - e^{\bar{C}_0\bar{b}} + \bar{C}_0\bar{b}e^{\bar{C}_0\bar{b}}}{(e^{\bar{C}_0\bar{b}} - 1)^2} \bar{w}_2(\bar{t}) \\ \quad + \bar{K}_{f2} \left(\frac{\bar{C}_0\bar{b}}{e^{\bar{C}_0\bar{b}} - 1} - 1 - \bar{C}_0\bar{Q} \right) \end{cases}. \quad (S8)$$

Section S2: Tables of system parameters

Table S1 Material properties and geometric parameters.

Parameter	Definition	Value	Unit
L_f	Length of LCE fiber	0.1	m

K_{f1}	Elastic coefficient of LCE fiber 1	0~15	N/m
K_{f2}	Elastic coefficient of LCE fiber 2	0~15	N/m
K_s	Spring factor	0~15	N/m
m	Mass of mass block	0.01	kg
C_0	Thermal contraction coefficient	-0.005~-0.001	1/°C
b	Gradient of heat flux	0~150	°C/m
τ	Thermal characteristic time	0~0.2	s
β_1	First damping coefficient	0~0.004	kg/s
β_2	Second damping coefficient	0~0.02	kg/m

Table S2 Dimensionless parameters

Parameter	$\bar{C}_0$	$\bar{K}_{f1}$	$\bar{K}_{f2}$	$\bar{K}_s$	$\bar{b}$	$\bar{\beta}_1$	$\bar{\beta}_2$	$\bar{\tau}$
Value	-0.5~-0.1	0~15	0~15	0~15	0~1.5	0~0.04	0~0.2	0~0.02

Section S3: Derivations of Eqs. (11)-(13) and stability analysis

(1) Derivations of Eqs. (11)-(13)

We introduce time variables $T_n = \varepsilon^n t$ (for $n=0,1,2$) representing different scales to discuss the approximate solutions. The solutions for Eq. (10) can be expressed as

$$\begin{cases} y_1 = y_{10}(T_0, T_1) + \varepsilon y_{11}(T_0, T_1) + \varepsilon^2 y_{12}(T_0, T_1) \\ y_2 = y_{20}(T_0, T_1) + \varepsilon y_{21}(T_0, T_1) + \varepsilon^2 y_{22}(T_0, T_1) \end{cases} \quad (S9)$$

$$\begin{cases} \dot{y}_1 = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 \\ \dot{y}_2 = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 \end{cases} \quad (S10)$$

$$\begin{cases} \ddot{y}_1 = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_1^2 + 2D_0 D_2) \\ \ddot{y}_2 = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_1^2 + 2D_0 D_2) \end{cases} \quad (S11)$$

where $D_n = \frac{\partial}{\partial T_n}$ (for $n=0,1,2$) denotes the symbol of partial differential operator.

Substituting Eqs. (S9)-(S11) into Eq. (10) and equating coefficients of like powers of ε gives the following system of linear partial differential equations

$$\begin{cases} D_0^2 y_{10} + \omega_1^2 y_{10} = 0 \\ D_0^2 y_{20} + \omega_2^2 y_{20} = 0 \end{cases} \quad (S12)$$

$$\begin{cases} D_0^2 y_{11} + \omega_1^2 y_{11} = -2D_0 D_1 y_{10} + \delta_1 (D_0 y_{10} - c_1 |D_0 y_{10}| D_0 y_{10}) + k(y_{20} - y_{10}) \\ D_0^2 y_{21} + \omega_2^2 y_{21} = -2D_0 D_1 y_{20} + \delta_2 (D_0 y_{20} - c_2 |D_0 y_{20}| D_0 y_{20}) + k(y_{10} - y_{20}) \end{cases} \quad (S13)$$

The solution to the zeroth-order approximation Eq. (S12) is expressed in complex-valued form as

$$\begin{cases} y_{10} = A_1(T_1)e^{i\omega_1 T_0} + \bar{A}_1(T_1)e^{-i\omega_1 T_0} \\ y_{20} = A_2(T_1)e^{i\omega_2 T_0} + \bar{A}_2(T_1)e^{-i\omega_2 T_0} \end{cases}, \quad (S14)$$

where A_i (for $i=1,2$) is a complex-valued function to be determined, and $\bar{A}_i$ (for $i=1,2$) denotes its complex conjugate. Expressing A_i in exponential form as

$$\begin{cases} A_1(T_1) = \frac{1}{2} a_1(T_1) e^{i\phi_1(T_1)} \\ A_2(T_1) = \frac{1}{2} a_2(T_1) e^{i\phi_2(T_1)} \end{cases}, \quad (S15)$$

where both $a_i(T_1)$ and $\phi_i(T_1)$ (for $i=1,2$) are real-valued functions of time.

Substituting Eq. (S14) into the right-hand side of the first-order approximation Eq. (S13) yields

$$\begin{cases} D_0^2 y_{11} + \omega_1^2 y_{11} = -2i\omega_1 D_1 A_1 e^{i\omega_1 T_0} + 2i\omega_1 D_1 \bar{A}_1 e^{-i\omega_1 T_0} + i\delta_1 \omega_1 A_1 e^{i\omega_1 T_0} - i\delta_1 \omega_1 \bar{A}_1 e^{-i\omega_1 T_0} \\ \quad - i\delta_1 \frac{4a_1^2 \omega_1^2 c_1}{3\pi} e^{i(\omega_1 T_0 + \phi_1)} + i\delta_1 \frac{4a_1^2 \omega_1^2 c_1}{3\pi} e^{-i(\omega_1 T_0 + \phi_1)} + \dots + k(A_2 e^{i\omega_2 T_0} - A_1 e^{i\omega_1 T_0}) \\ D_0^2 y_{21} + \omega_2^2 y_{21} = -2i\omega_2 D_1 A_2 e^{i\omega_2 T_0} + 2i\omega_2 D_1 \bar{A}_2 e^{-i\omega_2 T_0} + i\delta_2 \omega_2 A_2 e^{i\omega_2 T_0} - i\delta_2 \omega_2 \bar{A}_2 e^{-i\omega_2 T_0} \\ \quad - i\delta_2 \frac{4a_2^2 \omega_2^2 c_2}{3\pi} e^{i(\omega_2 T_0 + \phi_2)} + i\delta_2 \frac{4a_2^2 \omega_2^2 c_2}{3\pi} e^{-i(\omega_2 T_0 + \phi_2)} + \dots + k(A_1 e^{i\omega_1 T_0} - A_2 e^{i\omega_2 T_0}) \end{cases} \quad (S16)$$

To prevent the emergence of secular terms, the sum of the coefficients of $e^{i\omega_1 T_0}$ and $e^{i\omega_2 T_0}$ should vanish. Considering $\omega_2 = \omega_1 + \varepsilon\sigma$, the function A_i (for $i=1,2$) satisfies the following conditions

$$\begin{cases} -2i\omega_1 D_1 A_1 + i\delta_1 \omega_1 A_1 - i\delta_1 \frac{8a_1 \omega_1^2 c_1}{3\pi} A_1 + kA_2 e^{i\sigma T_1} - kA_1 = 0 \\ -2i\omega_2 D_1 A_2 + i\delta_2 \omega_2 A_2 - i\delta_2 \frac{8a_2 \omega_2^2 c_2}{3\pi} A_2 + kA_1 e^{-i\sigma T_1} - kA_2 = 0 \end{cases}, \quad (S17)$$

where σ is the frequency detuning parameter.

Through substituting Eq. (S15) into Eq. (S17), we obtain

$$\begin{cases} -\frac{da_1}{dT_1} - ia_1 \frac{d\phi_1}{dT_1} + \frac{\delta_1 a_1}{2} - \frac{4\delta_1 a_1^2 \omega_1 c_1}{3\pi} - i \frac{ka_2}{2\omega_1} e^{i\psi} + i \frac{ka_1}{2\omega_1} = 0 \\ -\frac{da_2}{dT_2} - ia_2 \frac{d\phi_2}{dT_2} + \frac{\delta_2 a_2}{2} - \frac{4\delta_2 a_2^2 \omega_2 c_2}{3\pi} - i \frac{ka_1}{2\omega_2} e^{-i\psi} + i \frac{ka_2}{2\omega_2} = 0 \end{cases}, \quad (\text{S18})$$

where the phase difference is $\psi = (\omega_2 t + \phi_2) - (\omega_1 t + \phi_1) = \phi_2 - \phi_1 + \sigma T_1$.

Then, separating the real and imaginary components of Eq. (S18) yields

$$\begin{cases} \frac{da_1}{dT_1} = \frac{\delta_1}{2} a_1 - \frac{4\delta_1 \omega_1 c_1}{3\pi} a_1^2 + \frac{k}{2\omega_1} a_2 \sin \psi, \\ \frac{d\phi_1}{dT_1} = \frac{k}{2\omega_1} - \frac{a_2}{a_1} \frac{k}{2\omega_1} \cos \psi \end{cases} \quad (\text{S19})$$

$$\begin{cases} \frac{da_2}{dT_2} = \frac{\delta_2}{2} a_2 - \frac{4\delta_2 \omega_2 c_2}{3\pi} a_2^2 - \frac{k}{2\omega_2} a_1 \sin \psi, \\ \frac{d\phi_2}{dT_2} = \frac{k}{2\omega_2} - \frac{a_1}{a_2} \frac{k}{2\omega_2} \cos \psi \end{cases} \quad (\text{S20})$$

Given $\psi = \phi_2 - \phi_1 + \sigma T_1$, we have

$$\begin{aligned} \frac{d\psi}{dT_1} &= \frac{d\phi_2}{dT_1} - \frac{d\phi_1}{dT_1} + \sigma \\ &= \frac{k}{2} \left(\frac{1}{\omega_2} - \frac{1}{\omega_1} \right) + \cos \psi \frac{k}{2} \left(\frac{1}{\omega_2} \frac{a_2}{a_1} - \frac{1}{\omega_1} \frac{a_1}{a_2} \right) + \sigma. \end{aligned} \quad (\text{S21})$$

Restoring Eqs. (S19)-(S21) from the T_1 scale to the t scale yields Eqs. (11)-(13).

(2) Stability analysis

Linearizing Eqs. (11)-(13) around the equilibrium point (a_1^*, a_2^*, ψ^*) yields the Jacobian matrix, defined as

$$J = \begin{bmatrix} \frac{\partial \dot{a}_1}{\partial a_1} & \frac{\partial \dot{a}_1}{\partial a_2} & \frac{\partial \dot{a}_1}{\partial \psi} \\ \frac{\partial \dot{a}_2}{\partial a_1} & \frac{\partial \dot{a}_2}{\partial a_2} & \frac{\partial \dot{a}_2}{\partial \psi} \\ \frac{\partial \dot{\psi}}{\partial a_1} & \frac{\partial \dot{\psi}}{\partial a_2} & \frac{\partial \dot{\psi}}{\partial \psi} \end{bmatrix}, \quad (\text{S22})$$

where each element is given by

$$\begin{aligned}
J_{11} &= \frac{\varepsilon_1}{2} - \frac{8\varepsilon_1\omega_1c_1}{3\pi}a_1^*, \quad J_{12} = \frac{\bar{K}_s}{2\omega_1}\sin\psi^*, \quad J_{13} = \frac{\bar{K}_s}{2\omega_1}a_2^*\cos\psi^*, \\
J_{21} &= -\frac{\bar{K}_s}{2\omega_2}\sin\psi^*, \quad J_{22} = \frac{\varepsilon_2}{2} - \frac{8\varepsilon_2\omega_2c_2}{3\pi}a_2^*, \quad J_{23} = -\frac{\bar{K}_s}{2\omega_2}a_1^*\cos\psi^*, \\
J_{31} &= -\frac{\bar{K}_s}{2}\cos\psi^*\left(\frac{a_2^*}{\omega_1a_1^{*2}} + \frac{1}{\omega_2a_2^*}\right), \quad J_{32} = \frac{\bar{K}_s}{2}\cos\psi^*\left(\frac{1}{\omega_1a_1^*} + \frac{a_1^*}{\omega_2a_2^{*2}}\right), \\
J_{33} &= -\frac{\bar{K}_s}{2}\sin\psi^*\left(\frac{a_2^*}{\omega_1a_1^*} - \frac{a_1^*}{\omega_2a_2^*}\right).
\end{aligned} \tag{S23}$$

The characteristic equation is

$$\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 = 0, \tag{S24}$$

where the coefficients are

$$\begin{aligned}
p_1 &= -(J_{11} + J_{22} + J_{33}), \\
p_2 &= J_{11}J_{22} + J_{11}J_{33} + J_{22}J_{33} - J_{12}J_{21} - J_{13}J_{31} - J_{23}J_{32}, \\
p_3 &= -[J_{11}(J_{22}J_{33} - J_{23}J_{32}) - J_{12}(J_{21}J_{33} - J_{23}J_{31}) + J_{13}(J_{21}J_{32} - J_{22}J_{31})].
\end{aligned} \tag{S25}$$

Given the Routh-Hurwitz criterion, the necessary and sufficient conditions for local asymptotic stability of an equilibrium point are $p_1 > 0$, $p_3 > 0$, and $p_1p_2 - p_3 > 0$. If all eigenvalues have negative real parts, the equilibrium point, i.e., the synchronous solution, is concluded to be stable; otherwise, it is unstable.

Section S4: Illustration of in-phase and anti-phase solutions in monostable and bistable regimes

In order to provide an intuitive illustration of the in-phase and anti-phase solutions in the monostable and bistable regimes, Fig. S1 plots the final phase difference as a function of $\bar{K}_s$ and $\Delta\omega$, where the other parameters are $\bar{K}_{f1}=10$, $\bar{C}_0=-0.3$, $\bar{b}=1$, $\bar{\beta}_1=0.01$, $\bar{\beta}_2=0.05$, and $\bar{\tau}=0.01$. As displayed in Fig. S1a, under a fixed detuning $\Delta\omega=0.3$ obtained by setting $\bar{K}_{f2}=11.84$, the system is in a monostable regime with the anti-phase solution at small $\bar{K}_s$. As $\bar{K}_s$ increases, it transitions into a bistable regime where both in-phase and anti-phase solutions coexist. In such cases, the initial phase condition influences both the route to synchronization and the final attractor to which the system converges. This represents a path-dependent behavior. In Fig. S1b, for a fixed coupling strength $\bar{K}_s=0.5$, a bistable regime (in-phase and anti-phase) exists at small detuning $\Delta\omega$. Notably, the range of $\Delta\omega$ from -0.3 to 0.3 is achieved by changing $\bar{K}_{f2}$ from 8.32 to 11.84. With increasing $\Delta\omega$, the in-phase solution first loses

stability, leading to a monostable regime of the anti-phase solution. Further increase in $\Delta\omega$ eventually results in a loss of synchronization.

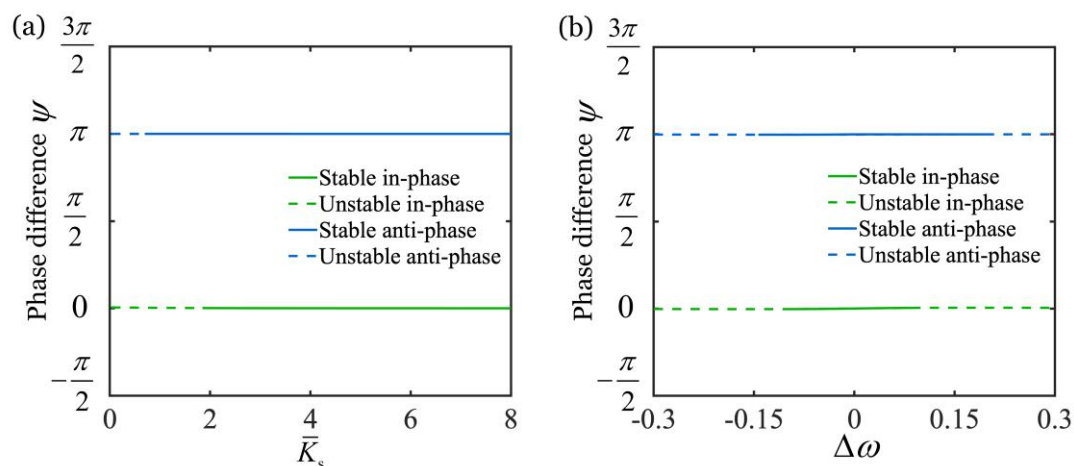


Fig. S1 Stability of phase-locking solutions. (a) Phase-locking solutions for different $\bar{K}_s$ under fixed $\Delta\omega = 0.3$. (b) Phase-locking solutions for different $\Delta\omega$ under fixed $\bar{K}_s = 0.5$. Bistability, where both in-phase and anti-phase locking solutions coexist, occurs for matched $\bar{K}_s$ and $\Delta\omega$. A decrease in $\bar{K}_s$ or an increase in $\Delta\omega$ can trigger a transition from bistable regime to monostable regime.